Lieb-Thirring inequalities for eigenvalues of Schrödinger and Dirac operators

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Schrödinger operators

Definition (Schrödinger operators)

Let

$$H = -\triangle + V(x) \quad \text{on } L^2(\mathbb{R}^d),$$

where $\triangle = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ denotes the Laplace operator in $\mathbb{R}^d$ and V is a real-valued potential function of $x \in \mathbb{R}^d$.

Example (Hydrogenic atom)

$$H = -\triangle - \frac{Z}{|x|} \quad \text{on } L^2(\mathbb{R}^d),$$

Our Goal

Energy eigenvalues for Schrödinger operators

$E_0, E_1, E_2, \ldots$ denote all non-positive eigenvalues of H.

The Riesz mean of order $\gamma \geq 0$

$$\sum_j |E_j|^\gamma$$

should be estimated in the view of V. In particular,

$$\sum_j |E_j|^\gamma = \begin{cases} \text{the number of eigenvalues} & \text{for } \gamma = 0 \\ \text{the sum of possible energies} & \text{for } \gamma = 1. \end{cases}$$

Lieb-Thirring inequalities

Theorem (Lieb and Thirring, 1976)

Let $\gamma \geq 0$*. Assume that* $V_-(x) = \max\{-V(x), 0\}$ *satisfies the condition* $V_- \in L^{\gamma+d/2}(\mathbb{R}^d)$*. Then, there is* $L_{\gamma,d} > 0$ *which is independent of* V *such that*

$$\sum_j |E_j|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-(x)^{\gamma+d/2} \, dx \tag{1}$$

holds when $\gamma \geq 1/2$ *for* $d = 1$*,* $\gamma > 0$ *for* $d = 2$*, and* $\gamma \geq 0$ *for* $d \geq 3$*. Otherwise, there is* V *that violates the inequalities (4) for any finite choice of* $L_{\gamma,d}$*.*

Remarks on Lieb-Thirring inequalities

- Cwikel [1], Lieb [7] and Rozenbljum [10]; the critical case $\gamma = 0$ for $d \geq 3$. (1970s)
- Lieb and Thirring [9]; almost all cases in Lieb-Thirring inequalities. (1976)
- Weidl [13]; the remaining case $\gamma = 1/2$ for $d = 1$. (1996)

Semi-classical approximation

The coefficients $L_{\gamma,d}$ should be compared to classical ones obtained by the semi-classical approximation

$$L^{\mathrm{cl}}_{\gamma,d} = (2\pi)^d \int_{|p|\le 1} (1-|p|^2)^\gamma \, dp = \frac{\Gamma(\gamma+1)}{(4\pi)^{d/2}\Gamma(\gamma+1+d/2)}$$

where Γ is the Gamma function.
It is also known that $L_{\gamma,d}/L^{\mathrm{cl}}_{\gamma,d} \ge 1$ for all possible γ and d, and that $L_{\gamma,d}/L^{\mathrm{cl}}_{\gamma,d}$ is non-increasing on γ. (Aizenman and Lieb, 1978)

Semi-classical approximation

- Helffer and Robert [3]; $L_{\gamma,d}/L^{\mathrm{cl}}_{\gamma,d} > 1$ for $\gamma < 1$. (1990)
- Hundertmark, Lieb and Thomas [5]; $L_{1/2,1} = 2L^{\mathrm{cl}}_{1/2,1}$. (1998)
- Hundertmark, Laptev and Weidl [4]; $L_{\gamma,1} = L^{\mathrm{cl}}_{\gamma,1}$ for $\gamma \geq 3/2$. (2000)
- Laptev and Weidl [6] enables that $L_{\gamma,d} = L^{\mathrm{cl}}_{\gamma,d}$ if $\gamma \geq 3/2$ for all d. (2000)

Recent results (1)

Dolbeault, Laptev and Loss [2] have improved the coefficient which is also known as best possible at the present time.

Dolbeault, Laptev and Loss, 2008

$$L_{1,d}/L^{\mathrm{cl}}_{1,d} \leq \pi/\sqrt{3} = 1.81... \quad \text{for all } d.$$

[2] J. Dolbeault, A. Laptev and M. Loss; J. Eur. Math. Soc., Vol. 10, pp. 1121-1126. (2008)

Recent results (2)

Rumin [11] and Solovej [12] has proposed a new approach of proving that

Rumin and Solovej, 2011

$$L_{1,d}/L_{1,d}^{\mathrm{cl}} \leq \left(\frac{d+4}{d}\right)^{d/2}.$$

[11] M. Rumin, Duke Math. J., **160**, no. 3, 567–597. (2011)
[12] J. P. Solovej,
"The Lieb-Thirring inequality." (2011)

Lieb-Thirring conjecture

It is conjectured by Lieb and Thirring [9] that the optimal $L_{1,3}$ coincides with $L_{1,3}^{\mathrm{cl}}$.

Conjecture (Lieb and Thirring, 1976)

$$L_{1,3} = L_{1,3}^{cl}$$

[9] E. H. Lieb and W. E. Thirring, "Inequalities for the Moments of the Eigenvalues of the Schrödinger Hamiltonian and their Relation to Sobolev Inequarities," in *Studies in Mathematical Physics* (1976), pp. 269–303.

Stability of Matter

$$H = \frac{1}{2}\sum_{j=1}^{N}\left(-i\nabla_j + \sqrt{\alpha}A(x_j)\right)^2 + \alpha V(X,R),$$

where $\alpha = e^2/\hbar c \approx 1/137 > 0$ is Sommerfeld's fine structure constant, A is an arbitrary magnetic vector potential in $L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$. The Coulomb potential is written by

$$V(X,R) = \sum_{j=1}^{N}\sum_{k=j+1}^{N}\frac{1}{|x_j - x_k|} - \sum_{j=1}^{N}\sum_{k=1}^{M}\frac{Z_k}{|x_j - R_k|} + \sum_{j=1}^{M}\sum_{k=j+1}^{M}\frac{Z_j Z_k}{|R_j - R_k|},$$

where M is number of nuclei.

Stability of Matter

Why is our world stable?

Lieb and Thirring [9] have improved the result by Dyson and Lenard for the stability of non-relativistic matter.

Theorem (Stability of matter of the second kind)

Let $Z_{max} = \max_j\{Z_j\}$. For all normalized, antisymmetric wavefunction ψ with q spin states,

$$(\psi, H\psi) \geq -0.231\alpha^2 Nq^{2/3}\left(1 + 2.16 Z_{max}(M/N)^{1/3}\right)^2.$$

It should be remarked that $N^{1/3}M^{2/3} \leq N + M$ implies the linear dependence on the total number of particles.

Pseudo-relativistic operators

Definition (Pseudo-relativistic operators)

Let

$$H = |-i\nabla| + V(x) \quad \text{on } L^2(\mathbb{R}^d),$$

where ∇ denotes the differential vector in $\mathbb{R}^d$
and V is a real-valued potential function of $x \in \mathbb{R}^d$.

It should be noted for relativistic mechanics that it is necessary to consider the operators

$$T = \sqrt{\mathbf{p}^2 + \mu^2} - \mu \tag{2}$$

for $\mu > 0$, where

$$\sqrt{-\triangle} - \mu \leq \sqrt{\mathbf{p}^2 + \mu^2} - \mu \leq \sqrt{-\triangle}. \tag{3}$$

LT inequalities for pseudo-relativistic operators

Theorem (Lieb 1980, and Daubechies 1983)

Let $\gamma \geq 0$. Assume that $V_-(x) = \max\{-V(x), 0\}$ satisfies the condition $V_- \in L^{\gamma+d}(\mathbb{R}^d)$. Then, there is $L_{\gamma,d} > 0$ which is independent of V such that

$$\sum_j |E_j|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-(x)^{\gamma+d} \, dx \qquad (4)$$

holds when $\gamma > 0$ for $d = 1$, and $\gamma \geq 0$ for $d \geq 2$.

Dirac operators

Definition (Dirac operators, $d = 3$)

Let

$$H_D = \alpha \cdot \mathbf{p} + \beta + V(x) \quad \text{on } L^2(\mathbb{R}^3),$$

where $\mathbf{p} = -i\nabla$ denotes the momentum operator in $\mathbb{R}^3$ and V is a real-valued potential function of $x \in \mathbb{R}^3$. Here,

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \tag{5}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the three Pauli matrices in standard representation:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

Lieb-Thirring inequalities for Dirac operators (1)

Theorem (Cancelier, Lévy-Bruhl and Nourrhgat, 1995)

Let $\gamma \geq 0$. Then, there is $C_\gamma > 0$ which is independent of V such that

$$\sum_j (1 - E_j^2)^\gamma \leq C_\gamma \int_{\mathbb{R}^d} |V(x)|^\gamma (|V(x)|^3 + |V(x)|^{3/2}) \, dx. \quad (7)$$

The proof of the Theorem follows from

$$\sum_j (1 - E_j^2)^\gamma = \gamma \int_0^1 \lambda^{\gamma-1} N(E, V) \, dE, \quad (8)$$

where $N(E, V)$ is the number of eigenvalues of H_D in $(-\sqrt{1-E}, \sqrt{1-E})$.

The number of eigenvalues for Dirac operators (2)

$$V \in L^p(\mathbb{R}^3) + \varepsilon L^\infty(\mathbb{R}^3) \Leftrightarrow \int_{|V(x)| \geq \varepsilon} |V(x)|^p \, dx < \infty \tag{9}$$

Theorem (Cancelier, Lévy-Bruhl and Nourrhgat, 1995)

Assume that $V(x)$ satisfies the condition $V \in L^3(\mathbb{R}^3) + \frac{E}{4} L^\infty(\mathbb{R}^3)$. Then, there is $C > 0$ which is independent of V such that

$$N(E, V) \leq C \int_{|V(x)| \geq E/4} (|V(x)|^3 + |V(x)|^{3/2}) \, dx. \tag{10}$$

The finite number of eigenvalues

Corollary (Cancelier, Lévy-Bruhl and Nourrhgat, 1995)

If $V \in L^3(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$, the number $N(V)$ of eigenvalues of H_D in $(-1, 1)$ is finite, and there is $C > 0$ which is independent of V such that

$$N(V) \leq C \int_{\mathbb{R}^d} (|V(x)|^3 + |V(x)|^{3/2})\, dx. \tag{11}$$

Concluding remarks

- We reviewed the topics around Lieb-Thirring inequalities for Schrödinger operators, pseudo-relativistic operators and Dirac operators.
- We share the proof of Lieb-Thirring inequalities for Dirac operators with rough value of coefficients..
- Lieb-Thirring conjecture remains open at the present time.