Research Review

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26 Sep. 2016

[1] On leave from Matsuyama University.

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Cauchy problem

Let us consider the Cauchy problem for the Schrödinger equation

$$\begin{cases} i\dfrac{\partial}{\partial t}u(t,x) = -\dfrac{1}{2}\triangle u(t,x) \\ u(0,x) = u_0(x) \end{cases} \tag{1}$$

where u is a complex-valued function of $(t,x) = (t,x_1,\dots,x_n) \in \mathbb{R}\times\mathbb{R}^n$, $i=\sqrt{-1}$ and $\triangle$ is Laplacian in $\mathbb{R}^n$.

Decay rate of wavefunctions

Put $u_0(x) = e^{-\frac{1}{2}|x|^2}$. Then, we can explicitly calculate

$$u(t,x) = (1+it)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2(1+it)}} \tag{2}$$

by virtue of Cauchy's integral theorem. From this expression, we have the estimate

$$(1+t^2)^{-\frac{n}{4}} e^{-\frac{1}{2}|x|^2} \le |u(t,x)| \le (1+t^2)^{-\frac{n}{4}} \tag{3}$$

for any $(t,x) \in \mathbb{R} \times \mathbb{R}^n$. Therefore, it becomes clear that $|u(t,x)|$ should behave $|t|^{-\frac{n}{2}}$ asymptotically as $t \to \pm\infty$ for every fixed $x \in \mathbb{R}^n$.

Polynomial time decay, example

Theorem (Dan (2011) for $n = 1$)

Let $xu_0 \in L^2(\mathbb{R})$. *If* $\xi^{-2}\hat{u}_0(\xi) \in L^2(|\xi| < 1)$ *and* $\xi^{-1}(\partial_\xi \hat{u}_0)(\xi) \in L^2(|\xi| < 1)$, *then*

$$|u(t,x)| \leq C(1 + |x|)|t|^{-1}, \tag{4}$$

for any $t \in \mathbb{R}$ *and* $x \in \mathbb{R}$, *where* $C > 0$ *is a constant which is independent both of* t *and* x.

See also

- M. Miyamoto, J. Phys. A: Math. Gen., Vol. 35, pp. 7159-7171. (2002)
- J. Rauch, Commun. math. Phys., Vol. 61, pp. 149-168. (1978)

Exponential time decay

Using the Fourier integral operator

$$I_\varphi v(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\varphi(x,\xi)} \hat{v}(\xi)\, d\xi \qquad (5)$$

with $\varphi(x,\xi) = x \cdot \xi - i\mu x \cdot \frac{\xi}{|\xi|}$.

Theorem (D and K. Kajitani (2002))

Let $v_0 \in H^s$ with $s = [n/2] + 1$. If $u_0 = I_\varphi v_0$, $Re\mu > 0$ and $Im\mu > 0$, then for any $\delta > 0$ there is $C > 0$ such that

$$|u(t,x)| \leq C\|v_0\|_s e^{-Re\mu Im\mu t + (Re\mu+\delta)\langle x\rangle} \qquad (6)$$

for any $t \geq 0$ and $x \in \mathbb{R}^n$.

Exponential time decay, example

Example

As an example of Theorem 2, we can give the function $v_0(x) = e^{-x^2}$ when $n = 1$. Then, $u_0 = I_\varphi v_0$ is calculated as

$$u_0(x) = \frac{1}{2} e^{-x^2} \left\{ e^{\mu x} + e^{-\mu x} + \frac{i}{\sqrt{2\pi}} (e^{\mu x} - e^{-\mu x}) \int_0^x e^{y^2} \, dy \right\}, \tag{7}$$

which gives an example of an initial value that the solution u to the Cauchy problem (1) has the property of exponential time decay.

Cauchy problem in the weighted Sobolev spaces

The Cauchy problem (1) in the weighted Sobolev spaces is equivalent to the Cauchy problem

$$\begin{cases} i\dfrac{\partial}{\partial t}v(t,x) = \left(-\dfrac{1}{2}\triangle - i\sigma \displaystyle\sum_{j=1}^{n} \dfrac{x_j}{\langle x\rangle} D_j + c(x)\right) v(t,x) \\ v(0,x) = v_0(x) \end{cases} \tag{8}$$

in the usual Sobolev spaces, where $D_j = \dfrac{1}{i}\dfrac{\partial}{\partial x_j}$ and

$$c(x) = -\frac{1}{2}\left\{\sigma^2 - \frac{\sigma}{\langle x\rangle} - \frac{\sigma^2}{\langle x\rangle^2} + \frac{(n+1)\sigma}{\langle x\rangle^3}\right\}. \tag{9}$$

Uniqueness for the free Schrödinger operator

Let H^s_σ denote the weighted Sobolev space, which is defined by

$$H^s_\sigma = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n);\ e^{-\sigma\langle x\rangle} u \in H^s\} \tag{10}$$

for $\sigma \in \mathbb{R}$ as a subset of locally square-integrable functions in $\mathbb{R}^n$.

Theorem (Dan (2005))

Let $\sigma > 0$. If there is u in $C^0(\mathbb{R}; H^s_\sigma) \cap C^1(\mathbb{R}; H^{s-2}_\sigma)$ satisfying the Cauchy problem (1) with $u_0 = 0$, then $u = 0$ in the topology of $C^0(\mathbb{R}; H^s_\sigma) \cap C^1(\mathbb{R}; H^{s-2}_\sigma)$.

Uniqueness for variable coefficient operators

Let us consider

$$\begin{cases} i\dfrac{\partial}{\partial t}u(t,x) = (a(x,D) + b(x,D))\,u(t,x) \\ u(0,x) = u_0(x) \end{cases} \tag{11}$$

where
$a(x,D) = \sum_{j,k=1}^{n} D_j a_{jk}(x) D_k$, $b(x,D) = \sum_{j=1}^{n} b_j(x) D_j + c(x)$.
The coefficients $a_{jk}(x)$ and $b_j(x)$ are real-valued multiplication operators, while $c(x)$ is a complex-valued one. We assume $a_{kj} = a_{jk}$ and that there is a positive constant c_0 such that

$$\sum_{j,k} a_{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2 \tag{12}$$

for any $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$.

Uniqueness for variable coefficient operators, result

Moreover, we impose analiticity on the coefficients. To be precise, we introduce a function class. By $a \in \mathcal{A}$ we mean that there are positive constants C and ρ such that

$$\sup_{x \in \mathbb{R}^n} |D_x^\alpha a(x)| \leq C\rho^{-|\alpha|}|\alpha|! \tag{13}$$

for any $\alpha \in \mathbb{N}^n$. Then, $a_{jk} \in \mathcal{A}$, $b_j \in \mathcal{A}$ and $c \in \mathcal{A}$ are our last but essential assumption on the coefficients.

Theorem (Dan (2008))

Let $\sigma > 0$. If there exists $u \in C^0(\mathbb{R}; H_\sigma^s) \cap C^1(\mathbb{R}; H_\sigma^{s-2})$ satisfying the Cauchy problem (11) with $u_0 = 0$, then $u = 0$ in the topology of $C^0(\mathbb{R}; H_\sigma^s) \cap C^1(\mathbb{R}; H_\sigma^{s-2})$.

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