7.6.2 The $N^{\frac{7}{5}}$ Law

Bogolubov (1947) $\longrightarrow$ Bardeen-Cooper-Schrieffer (1957)

└ $a^\dagger$, a on $\mathcal{F}$

↓

Foldy (1961)

└ jellium model, i.e. free bosons in background.

$$\frac{E_0}{N} \simeq -I_0 \alpha^{\frac{5}{4}} \rho^{\frac{1}{4}}$$

for large ρ (conjectured)

$$E_0 \underset{\sim}{\overset{\leq}{\geq}} -CN^{\frac{7}{5}}$$

Dyson (1967)

"rigorous" Conlon-Lieb-Yau (1988)

Lieb-Solovej ~~(2004)~~

One-component (2001)

$\exists C_{\text{lower}} \leq C$, Two-component (2004)

$C \leq \exists C_{\text{upper}}$ Solovej (2006)

diameter $N^{-\frac{1}{3}}$

$N \gg 1$

$$H = -\frac{1}{2}\sum_{i=1}^{N} \Delta_i + \sum_{1 \leq i < j \leq N} \frac{e_i e_j}{|x_i - x_j|}$$

$$e_i = \pm 1$$

$$\inf_{\psi = \text{sym.}} (\psi, H\psi) = \inf_{\substack{\psi \in \otimes^N \mathcal{H} \\ \{e_i = \pm 1\}}} (\psi, H\psi)$$

Find ρ that minimizes

exchange-correlation energy

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla\sqrt{\rho(x)}|^2 dx - \underbrace{I_0 \alpha^{\frac{5}{4}} \int_{\mathbb{R}^3} \rho(x)^{\frac{5}{4}} dx}_{\times\, 2 I_0 \alpha^{\frac{5}{4}} \int \left(\frac{\rho(x)}{2}\right)^{\frac{5}{4}} dx}$$

under $\int_{\mathbb{R}^3} \rho(x)\, dx = N$

$$I_0 = \left(\frac{2}{\pi}\right)^{\frac{3}{4}} \int_0^\infty \left(1 + t^4 - t^2\sqrt{t^4+2}\right) dt$$

$$= \frac{2^{\frac{3}{2}}\,\Gamma\left(\frac{3}{4}\right)}{\cancel{4}\,5\,\pi^{\frac{1}{4}}\,\Gamma\left(\frac{1}{4}\right)} \approx 0.5744$$

$$\Phi(x) = N^{-\frac{4}{5}} \rho(N^{-\frac{1}{5}} x)^{\frac{1}{2}}$$

$$\Rightarrow \int_{\mathbb{R}^3} \Phi(x)^2 dx = N^{-\frac{\cancel{8}\,5}{5}} \int \rho(\underbrace{N^{-\frac{1}{5}} x}_{x'}) \underbrace{N^{-\frac{3}{5}} dx}_{dx'}$$

$$= 1.$$

Lane-Emden equation

$$-\Delta \Phi(x) \cancel{+}\, \underline{-}\, \frac{5}{2} I_0 \alpha^{\frac{5}{4}} \Phi(x)^{\frac{3}{2}} + \mu \Phi(x) = 0$$

$\mu > 0$

jellium $\xrightarrow{\text{easier}}$ two-component

derived (correct exponents in the N dependence)

Foldy

$$S = \frac{2}{\pi} 3^{\frac{1}{4}} \int_0^\infty \xi^2 (\underset{2}{4} + \xi^4)^{\frac{1}{2}} - \xi^2 - \underset{1}{2} \quad d\xi$$

$$= -1.606.$$

0.803

$$\frac{2}{\pi} \cancel{3}^{\frac{1}{4}} \cdot \left(\frac{4\pi}{\cancel{3}}\right)^{\frac{1}{4}}$$

$$= \frac{2^{\frac{3}{4}}}{\pi^{\frac{3}{4}}} = \left(\frac{2}{\pi}\right)^{\frac{3}{4}}$$

$$S = \frac{32}{5\pi} \left(\frac{3}{4}\right)^{\frac{1}{4}} \left\{ F\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right) - 2K\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right) \right\}$$

first

second.

complete elliptic integrals

$$= \frac{\pi}{2} \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} \left(\frac{1}{\sqrt{2}}\right)^{2m}$$

$$\left(\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\frac{1}{\sqrt{2}}t^2)}} \right.$$

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{E\left(\frac{1}{\sqrt{2}}\right)}{\frac{1}{2\sqrt{2}}} - \sqrt{2} K\left(\frac{1}{\sqrt{2}}\right)$$

Conlon-Lieb-Yau (1988)

$$E_0 \le -AN^{\frac{7}{5}} \quad \text{(Dyson)}$$

@ $E_0 \ge -BN^{\frac{7}{5}}$

one-component jellium model.

$$E_0 \le -CN\rho^{\frac{1}{4}} \quad \text{(Foldy, Girardeau)}$$

1961, 1962

@ $E_0 \ge -DN\rho^{\frac{1}{4}}$

$$H^J = -\frac{1}{2}\sum_i \left(\Delta_i + V(x_i)\right) + \sum |x_i - x_j|^{-1} + \frac{1}{2}\rho_B \int_\Lambda V(x)\,dx$$

$$V(x) = \rho_B \int_\Lambda |x-y|^{-1}\,dy$$

Complete elliptic integrals

1st kind

$$K(k) = \int_0^1 \frac{dt}{(1-t^2)(1-k^2t^2)}$$

2nd kind

$$E(k) = \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt$$

$$\int_0^\infty 1+t^4 - \sqrt{(1+t^4)^2-1} \quad dt$$

$$1+t^4 = \frac{1}{1-s} \qquad 4t^3\,dt = \frac{1}{(1-s)^2}\,ds$$

$$= \int_0^1 \frac{1}{1-s} - \sqrt{\left(\frac{1}{1-s}\right)^2 - 1} \cdot \frac{1}{4}\, t^{-3} \frac{1}{(1-s)^2}\,ds$$

$$\left(t^4 = \frac{1}{1-s} - 1 = \frac{s}{1-s} \right)$$

$$= \frac{1}{4}\int_0^1 \left(\frac{1}{1-s} - \frac{\sqrt{\overset{2s-s^2}{1-(1-s)^2}}}{1-s} \right) \left(\frac{s}{1-s}\right)^{-\frac{3}{4}} \frac{1}{(1-s)^2}\,ds$$

$s(2-s)$

$$-\frac{1}{2}\Delta\sqrt{\rho} + -I_0^{\frac{5}{4}}\sqrt{\rho}^{\frac{3}{2}}$$

$$\left(-I_0\alpha^{\frac{5}{4}}(\rho^{\frac{1}{2}})^{\frac{5}{2}}\right)'$$

$$-\frac{5}{2}I_0\alpha^{\frac{5}{4}}(\rho^{\frac{1}{2}})^{\frac{3}{2}}$$

$$\left(\frac{1}{2}\left|\nabla \cancel{\sqrt{\ }}\rho^{\frac{1}{2}}\right|^2\right)' = \nabla\rho^{\frac{1}{2}}$$

$$\frac{1}{2}\int|\nabla\sqrt{\rho}|^2dx - I_0\alpha^{\frac{5}{4}}\int\sqrt{\rho}^{\frac{5}{2}}dx$$

$$=\int\underbrace{\left(\frac{1}{2}|\nabla\sqrt{\rho}|^2 - I_0\alpha^{\frac{5}{4}}\sqrt{\rho}^{\frac{5}{2}}\right)}dx$$

$\cancel{|\nabla\sqrt{\rho}|}$ $\quad \partial\left(\ \right)$ $\quad \frac{5}{2}I_0\alpha^{\frac{5}{4}}\sqrt{\rho}^{\frac{3}{2}}$

$$|\nabla\Phi|^2$$

$$=(\partial_{x^1}\Phi)^2+(\partial_{x^2}\Phi)^2+(\partial_{x^3}\Phi)^2$$

$\longrightarrow$

$$\Phi(x) = N^{-\frac{4}{5}} \rho(N^{-\frac{1}{5}}x)^{\frac{1}{2}}$$

$$-\Delta\Phi(x) = -N^{-\frac{5}{4}} \Delta\left(\rho(N^{-\frac{1}{5}}x)^{\frac{1}{2}}\right)$$

$$= -N^{-\frac{5}{4}-\frac{2}{5}} \Delta\rho(N^{-\frac{1}{5}}x)^{\frac{1}{2}}$$

$$\frac{25-8}{20} \quad \frac{17}{20}$$

$$-\frac{5}{2}$$

$$\frac{1}{2}\int |\nabla\sqrt{\rho(x)}|^2 dx$$

$$= \frac{1}{2}\left(\nabla\sqrt{\rho(x)}, \nabla\sqrt{\rho(x)}\right)$$

$$= -\frac{1}{2}\left(\sqrt{\rho(x)}, \Delta\sqrt{\rho(x)}\right)$$

$$= \left(\sqrt{\rho(x)}, -\frac{1}{2}\Delta\sqrt{\rho(x)}\right)$$

$$-I_0\alpha^{\frac{5}{4}}\int_{\mathbb{R}^3} \rho(x)^{\frac{5}{4}} dx$$

$$= \left(\sqrt{\rho(x)}, -I_0\alpha^{\frac{5}{4}}\sqrt{\rho(x)}^{\frac{3}{\cancel{4}2}}\right)$$

$$\exists\rho \text{ s.t. } \underset{(\psi, H\psi)}{E_0} = -\mu$$

$$\Longrightarrow \quad -\Delta\Phi(x) - \frac{5}{2}I_0\alpha^{\frac{5}{4}}\Phi^{\frac{3}{2}} + \mu\Phi(x) = 0$$

$$T\varphi \sim N \times \text{diameter}^{-2}$$

$$N^{\frac{7}{5}} \Rightarrow \text{diameter} \sim N^{-\frac{1}{5}}$$