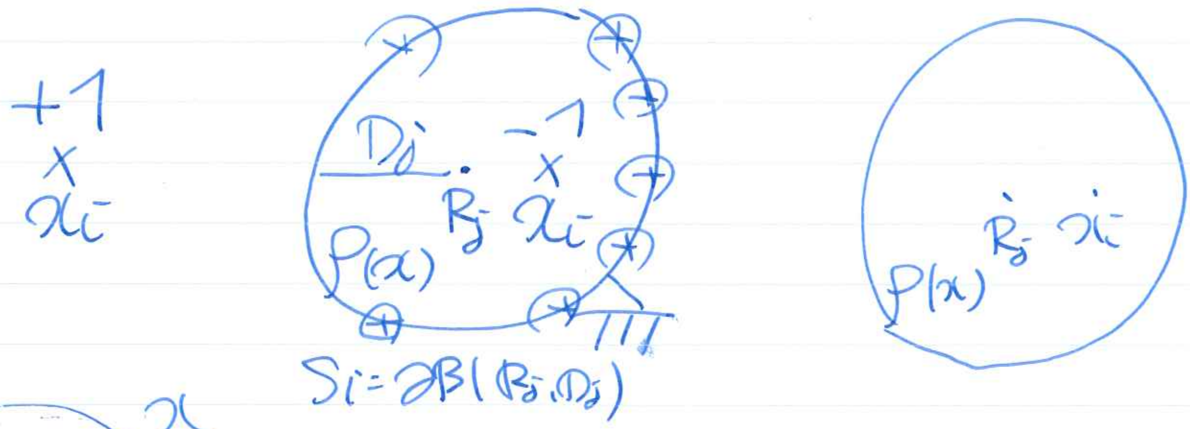
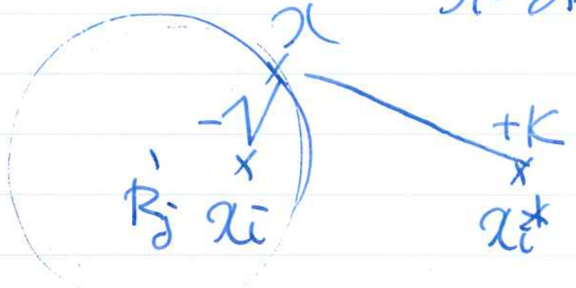
Scenario

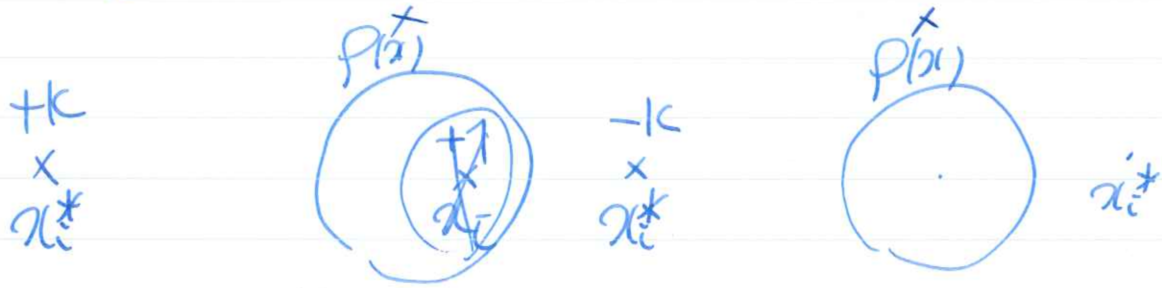




① $|x-x_i^*| = K|x-x_i|, \ K>1 \Longrightarrow \quad K = \frac{D_j}{|x_i-R_j|}$

② $\rho(x) = -\frac{1}{4\pi}\nabla\left[\frac{1}{|x-x_i|} - \frac{K}{|x-x_i^*|}\right]\cdot\hat{n}$

$\Longrightarrow \int_{S_i}\rho(x)\,dx = 1.$



③ $\Phi(x) = \int\frac{\nu_i(dy)}{|x-y|} \Longleftrightarrow -\Delta\Phi = 4\pi\nu_i$ in D

$$\int f(x)\,\nu_i(dx) = \cancel{\frac{1}{4\pi}}\int_{S_i} f(x)\,\rho(x)\,dx$$

$$\nu_i(dx) = \rho(x)\,\delta(|x-R_j|-D_j)\,dx$$

$$\frac{1}{|x-x_i|} = \frac{K}{|x-x_i^*|} \quad (K>1)$$

$$\Rightarrow \quad x_i^* = R_j + K^2(x_i - R_j), \quad K = \frac{D_j}{|x_i - R_j|}$$

$$\because) \quad |x - x_i^*|^2 = K^2 |x - x_i|^2$$

$$(K^2-1)|x|^2 - 2(K^2 x_i - x_i^*)\cdot x + K^2|x_i|^2 - |x_i^*|^2 = 0$$

$$(K^2-1)\left|x - \frac{K^2 x_i - x_i^*}{K^2-1}\right|^2 = \underbrace{\frac{(K^2 x_i - x_i^*)^2}{K^2-1} - K^2|x_i|^2 + |x_i^*|^2}_{(RHS)}$$

$$(LHS) \quad |x - R_j|^2 = D_j^2$$

$$\therefore \frac{K^2 x_i - x_i^*}{K^2-1} = R_j$$

$$\therefore x_i^* = R_j + K^2(x_i - R_j)$$

$$(RHS) = D_j^2$$

$$\Rightarrow \quad \cancel{K}\left(|x_i - R_j|^2 K^2 - D_j^2\right)(K^2-1) = 0$$

$$K^2 > 1 \Rightarrow \quad K = \frac{D_j}{|x_i - R_j|}.$$

$$\rho(x) = -\frac{1}{4\pi}\nabla\left[\frac{1}{|x-x_i|} - \overset{k}{\frac{D_j}{|x_i-R_j|}}\frac{1}{|x-x_i^*|}\right]\cdot\hat{n}$$

$$\Longrightarrow \int_{S_i} \rho(x)\,dx = 1.$$

$$\int \nu_i(dx)$$

$$\because)\ -\nabla[\cdots] = \frac{x-x_i}{|x-x_i|^3} - k\,\frac{x-x_i^*}{|x-x_i^*|^3}$$

$$\frac{1}{|x-x_i^*|} = \frac{1}{k}\frac{1}{|x-x_i|}$$

$$= \frac{x-x_i}{|x-x_i|^3} - k^2\,\frac{x-x_i^*}{|x-x_i|^3}$$

$$= \frac{1}{|x-x_i|^3}\left\{(x-x_i) - k^2(x-x_i^*)\right\}$$

$$x-x_i^* = x-R_j-k^2(x_i-R_j)$$

$$= \frac{1}{|x-x_i|^3}\left\{(x-x_i) - k^2(x-R_j) + (x_i-R_j)\right\}$$

$$(1-k^{-2})(x-R_j)$$

$$\therefore\ \rho(x) = \frac{1}{4\pi|x-x_i|^3}(1-k^{-2})(x-R_j)\cdot\frac{x-R_j}{|x-R_j|}$$

$$= \frac{D_j}{4\pi|x-x_i|^3}(1-k^{-2}) \qquad (\geq 0)$$

$$x = R_j + D_j\omega \qquad dx = D_j^2\, d\omega$$

$$\int_{S_i} \frac{dx}{|x - x_i|^3} = \int_{S_i} \frac{D_j^2\, d\omega}{|x_i - R_j - D_j\omega|^3}$$

$$= \int_{S_i} \frac{D_j^2\, d\omega}{\left[\, |x_i - R_j|^2 + D_j^2 - 2D_j (x_i - R_j)\cdot\omega \,\right]^{\frac{3}{2}}}$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, \frac{D_j^2}{\left[\, |x_i - R_j|^2 + D_j^2 - 2D_j |x_i - R_j| \cos\theta \,\right]^{\frac{3}{2}}}$$

$$-\cos\theta = s \qquad \sin\theta\, d\theta = ds$$

$$= 2\pi D_j^2 \int_{-1}^{1} \left[\, |x_i - R_j|^2 + D_j^2 + 2D_j |x_i - R_j|\, s \,\right]^{-\frac{3}{2}} ds$$

$$= 2\pi D_j^2 \cdot (-2) \frac{1}{2D_j |x_i - R_j|} \left[\, |x_i - R_j|^2 + D_j^2 + 2D_j |x_i - R_j| \,\right]^{-\frac{1}{2}} \Big|_{-1}^{1}$$

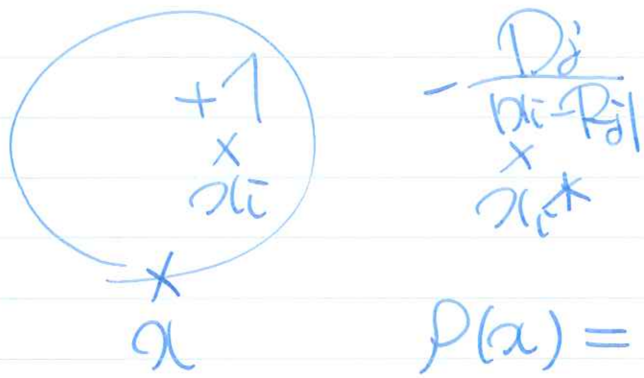
$$= \frac{2\pi D_j}{|x - R_j|} \underbrace{\left(\frac{1}{D_j - |x_i - R_j|} - \frac{1}{D_j + |x_i - R_j|} \right)}_{\frac{2|x - R_j|}{D_j^2 - |x_i - R_j|^2}}$$

$$= \frac{4\pi D_j}{D_j^2 - |x_i - R_j|^2}$$

$$\therefore \int_{S_i} \rho(x)\, dx = \frac{D_j}{4\pi} (1 - \kappa^{-2}) \frac{4\pi D_j}{D_j^2 - |x_i - R_j|^2} = 1 \;//$$

Scenario

$$\int_{S_i} \rho(x)\, d\omega = 1. \checkmark$$

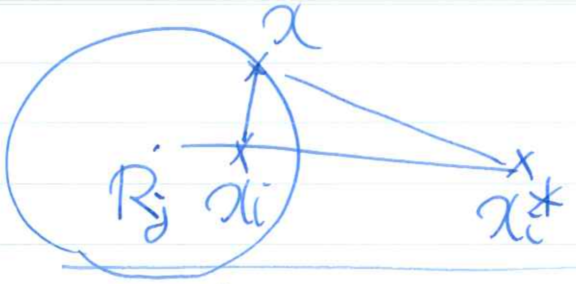


$$\rho(x) = \nabla\left(\frac{1}{|x-x_i|} - \Phi(x)\right)$$

$$\Phi(x) = \int \frac{\nu_i(dy)}{|x-y|}$$

$$x \in S_i \implies \Phi(x) = \frac{1}{|x-x_i|}$$

or $$\frac{D_j}{|x_i - R_j|} \frac{1}{|x-x_i^*|} = \frac{1}{|x-x_i|}$$



$$|x - x_i^*| = k|x - x_i|, \quad k > 1$$

$$\Longrightarrow \quad k = \frac{D_j}{|x_i - R_j|}$$

$$\because) \quad |x - x_i^*|^2 = k^2|x - x_i|^2$$

$$(\mathrm{LHS}) = |x|^2 + |x_i^*|^2 - 2x\cdot x_i^*$$

$$(\mathrm{RHS}) = k^2|x|^2 + k^2|x_i|^2 - 2k^2 x\cdot x_i$$

$$(k^2-1)|x|^2 + k^2|x_i|^2 - |x_i^*|^2 - 2(k^2 x_i - x_i^*)\cdot x = 0$$

$$(k^2-1)\left\{|x|^2 - \frac{2(k^2x_i - x_i^*)\cdot x}{k^2-1}\right\} = -k^2|x_i|^2 + |x_i^*|^2$$

$$(k^2-1)\left(x - \frac{k^2x_i - x_i^*}{k^2-1}\right)^2 = \frac{(k^2x_i - x_i^*)^2}{k^2-1} + k^2|x_i|^2 - |x_i^*|^2$$

$$\therefore \frac{k^2x_i - x_i^*}{k^2-1} = R_j$$

$$x_i^* = R_j + k^2(x_i - R_j)$$

$(D_j^2 - |R_j|^2)K^2$

$= D_j^2 - \cancel{|R_j|^2 + |R_j|^2}$

$D_j^2 = |R_j|^2 - \frac{1}{K^2-1}\{K^2|x_i|^2 - |R_j|^2 - 2K^2(x_i - R_j)\cdot R_j - K^4|x_i - R_j|^2\}$

$(D_j^2 - |R_j|^2)(K^2-1)$

$= -K^2|x_i|^2 + |R_j|^2 + 2K^2(x_i - R_j)\cdot R_j + K^4|x_i - R_j|^2$

$-D_j^2 + |R_j|^2 - |R_j|^2$

$= \left(-(D_j^2 - |R_j|^2) - |x_i|^2 + 2(x_i - R_j)\cdot R_j\right)K^2$

$+ |x_i - R_j|^2 K^4$

$= \left(-D_j^2 - |x_i|^2 + 2x_i \cdot R_j\right)K^2 + |x_i - R_j|^2 K^4$

$- x_i(x_i - R_j)$

$K^2 - \frac{D_j^2}{|x_i - R_j|^2}$ $\qquad K^2 > 1$

$(|x_i - R_j|^2 K^2 - D_j^2)(K^2 - 1) = 0$

$\underline{|x_i - R_j|^2 - D_j^2}$

$$P(x) = -\frac{1}{4\pi}\,\nabla\left[\frac{1}{|x-x_i|} - \frac{D_j}{|x_i-R_j|}\,\frac{1}{|x-x_i^*|}\right]\cdot\hat{n}$$

$$\Longrightarrow \int_{S_i} P(x)\, d_x = 1.$$

$$\because\ \nabla[\cdots] = \frac{x-x_i}{|x-x_i|^3} - \frac{D_j}{|x_i-R_j|}\,\frac{x-x_i^*}{|x-x_i^*|^3}$$

$$\frac{1}{|x-x_i^*|^3} = \left(\frac{|x_i-R_j|}{D_j}\,\frac{1}{|x-x_i|}\right)^3$$

$$= \frac{x-x_i}{|x-x_i|^3} - \frac{|x_i-R_j|^2}{D_j^2}\,\frac{x-x_i^*}{|x-x_i|^3}$$

$$= \frac{1}{|x-x_i|^3}\left\{(x-x_i) - \frac{|x_i-R_j|^2}{D_j^2}(x-x_i^*)\right\}$$

$$x-x_i^* = x-R_j - \frac{D_j^2}{|x_i-R_j|^2}(x_i-R_j)$$

$$= \frac{1}{|x-x_i|^3}\left\{(x-x_i) - \frac{|x_i-R_j|^2}{D_j^2}(x-R_j) + (x_i-R_j)\right\}$$

$$= \frac{1}{|x-x_i|^3}\left\{1 - \frac{|x_i-R_j|^2}{D_j^2}\right\}(x-R_j)$$

ii $$\rho(x) = \frac{1}{4\pi|x-x_i|^3}\left(1-\frac{|x_i-R_j|^2}{D_j^2}\right)(x-R_j)\cdot\frac{x-R_j}{|x-R_j|}$$

$$\frac{D_j^2}{D_j}$$

$$= \frac{1}{4\pi D_j|x-x_i|^3}\left(D_j^2-|x_i-R_j|^2\right)$$

$$x = R_j + D_j\omega$$

$$dx = D_j^2\, d\omega$$

$$\int_{S_i}\frac{d\omega}{|x-x_i|^3} = \int_{S_i}\frac{D_j^2\, d\omega}{|x_i-R_j-D_j\omega|^3}$$

$$= \int_{S_i}\frac{D_j^2\, d\omega}{\left[|x_i-R_j|^2+D_j^2-2D_j(x_i-R_j)\cdot\omega\right]^{\frac{3}{2}}}$$

$$= \int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta\, d\theta\,\frac{D_j^2}{[\qquad\qquad]^{\frac{3}{2}}}$$

$$|x_i-R_j|^2+D_j^2-2D_j|x_i-R_j|\cos\theta$$

$$-\cos\theta = s \qquad \sin\theta\, d\theta = ds$$

$$= 2\pi D_j^2\int_{-1}^{1}\left[|x_i-R_j|^2+D_j^2+2D_j|x_i-R_j|s\right]^{-\frac{3}{2}}ds$$

$$= 2\pi D_j^2\cdot(-\cancel{2})\frac{1}{\cancel{2}D_j|x_i-R_j|}\left[|x_i-R_j|^2+D_j^2+2D_j|x_i-R_j|s\right]^{-\frac{1}{2}}\Bigg|_{s=-1}^{s=1}$$

$$= \frac{2\pi D_j^2}{D_j|x_i-R_j|}\left(\frac{1}{D_j-|x_i-R_j|}-\frac{1}{D_j+|x_i-R_j|}\right)$$

$$\frac{2|x_i-R_j|}{D_j^2-|x_i-R_j|^2}$$

$$= \frac{4\pi \quad D_j^2}{D_j(D_j^2 - |x_i - P_j|^2)}$$

$$\therefore \int_{S_i} \rho(x)\, dw = \frac{1 \quad D_j^2}{4\pi D_j \cancel{\pi}} \cdot \frac{\cancel{4}\pi}{D_j} = \cancel{\frac{1}{D_j^2}}\ 1\ ?$$

$$x - x^* = x - R_j - \frac{D_j^2}{|x_i - R_j|^2}(x_i - R_j)$$

$$|x - x^*| = \frac{1}{|x_i - R_j|^2}\left| |x_i - R_j|^2 (x - R_j) - D_j^2 (x_i - R_j) \right|$$

$$|(a\cdot a)\, b - (b\cdot b)\, a| = |a|\cdot|b|\cdot|a-b|$$

$$\because)\ (\mathrm{LHS})^2 = |a|^4|b|^2 + |b|^4|a|^2 - 2|a|^2|b|^2 a\cdot b$$

$$= |a|^2|b|^2 \underbrace{\left(|a|^2 + |b|^2 - 2a\cdot b\right)}_{|a-b|^2} \quad //$$

$$= \frac{1}{|x_i - R_j|^{\cancel{2}}} \cancel{|x_i - R_j|} \cdot \cancel{|x - R_j|}^{D_j} \cdot |x - x_i|$$

$$= \frac{D_j}{|x_i - R_j|} |x - x_i|$$

$$\therefore \frac{1}{|x - x_i|} = \frac{D_j}{|x_i - R_j|} \frac{1}{|x - x^*|}$$

$$\text{if } |x - R_j| = D_j \quad //$$

$$\frac{|k^2 x_i - x_i^*|^2}{k^2-1} - k^2|x_i|^2 + |x_i^*|^2$$

$$\frac{|R_j|^2}{k^2-1} - k^2|x_i|^2 + |x_i^*|^2 = D_j^2$$

$$- \frac{}{k^2-1} - |x_i|^2$$

$$|x_i^*|^2 = |R_j - k^2(x_i - R_j)|^2$$

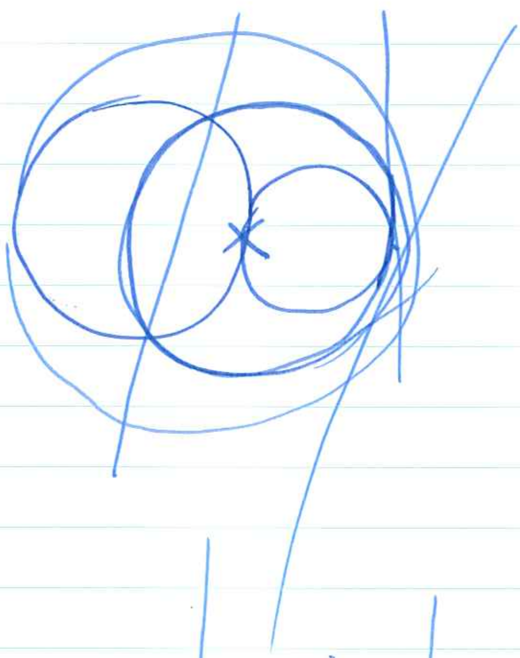
$$= |R_j|^2 + k^4|x_i - R_j|^2 - 2k^2 R_j \cdot (x_i - R_j)$$

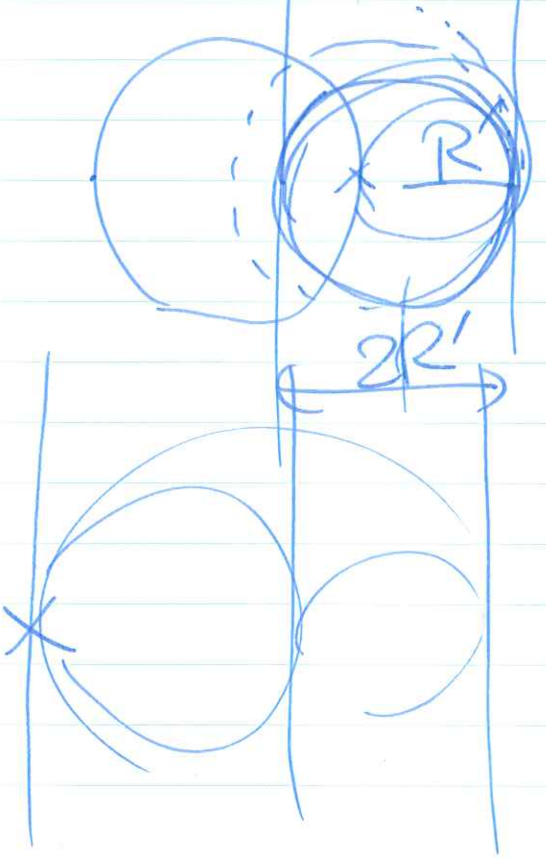
$$(|x_i - R_j|^2 k^2 - D_j)$$

$\lambda_4 \le \lambda_5 + \lambda_6$

$1 - \lambda_3 \le (1-\lambda_2) + (1-\lambda_1)$

$\lambda_1 + \lambda_2 \le \lambda_3 + 1$





$\cancel{\dfrac{r}{r_1}}$

$$\frac{1}{r_k} + \frac{1}{r_{n+k}} = \frac{4r(R')\left(\frac{1}{r} - \frac{1}{R}\right)}{(R' - r')^2}$$

$\gamma : \mathcal{H}^1 \to \mathcal{H}^1$

$\cdot\ \gamma u_j = \lambda_j u_j \quad j = 1, \dots, R = 6$

$\mathrm{tr}\,\gamma$

$\mathrm{Tr}\,\gamma = N = 3$

$\lambda_1 + \cdots + \lambda_6 = 3.$

$\exists\, u_1, u_2, u_3 \in \mathcal{H}^1.$

s.t. $\lambda_1 \geq \lambda_2 \geq \lambda_3 > 0$

$\lambda_1 + \lambda_2 + \lambda_3 = 1.$

$\lambda_1 = 1, \ \lambda_2 = 1. \ \lambda_3 = 1.$

$\psi = \mathcal{A}(u_1, u_2, u_3)$

$= \sum_{\pi \in S_3} (-1)^{|\pi|} u_1(x_{\pi(1)})\, u_2(x_{\pi(2)})\, u_3(x_{\pi(3)})$

$\psi : \mathcal{H}^3 \to \mathcal{H}^3$

$\Gamma_\psi \phi = (\psi, \phi)\psi$

$\Gamma_\psi(x, y) = \sum_j \lambda_j \overline{u(x)} u(y)$