Theorem 5.5
(Refined Electrostatic Inequality)

$0 < \lambda < 1$

$$W^\lambda : \mathbb{R}^3 = \bigcup_j \overline{\Gamma_j} \to \mathbb{R}$$

$$x \in \overline{\Gamma_j}$$

$$W^\lambda(x) = \frac{Z}{|x-R_j|} + \begin{cases} \frac{1}{2} D_j^{-1} \left(1 - D_j^{-2} |x - R_j|^2\right)^{-1} & |x-R_j| \leq \lambda D_j \\ \left(\sqrt{2Z} + \frac{1}{2}\right) |x-R_j|^{-1} & |x - R_j| > \lambda D_j \end{cases}$$

$$D_j = \frac{1}{2} \min_{i \neq j} |R_i - R_j|$$

$$\Longrightarrow U_c(\underline{X}, \underline{R}) \geq -\sum_{i=1}^{N} W^\lambda(x_i) + \frac{Z^2}{8} \sum_{j=1}^{M} \frac{1}{D_j}$$

pf. $x_i \in \Gamma_j$

Case 1. $|x_i - R_j| \le \lambda D_j$

$\exists! \nu_i$ s.t. $\operatorname{supp} \nu_i \subset S_i = \partial B(R_j, D_j)$

$S_i \subset \Gamma_j$

$$\int \frac{\nu_i(dy)}{|x-y|} = \begin{cases} \dfrac{1}{|x-x_i|} & |x-R_j| \ge D_j \\ \dfrac{D_j}{|x_i-R_j|} \dfrac{1}{|x-x_i^*|} & |x-R_j| < D_j \end{cases}$$

$$x_i^* - R_j = \frac{D_j^2}{|x_i-R_j|^2}(x_i - R_j)$$

$$\left(|x_i^* - R_j| = D_j \cdot \frac{D_j}{|x_i-R_j|} \ge D_j \right)$$

$$\therefore D(\nu_i, \nu_i) = \frac{1}{2}\int \frac{\nu_i(dx)}{|x-x_i|}$$

$$= \frac{D_j}{2|x_i-x_i^*|} \frac{1}{|x_i-x_i^*|}$$

$$x_i - x_i^* = x_i - R_j - \frac{D_j^2}{|x_i-R_j|^2}(x_i-R_j)$$

$$= -\left(\frac{D_j^2}{|x_i-R_j|^2} - 1 \right)(x_i - R_j)$$

$$= \frac{1}{2D_j} \cdot \frac{1}{1 - \frac{|x_i-R_j|^2}{D_j^2}}.$$

Case 2. $|x_i - R_j| > \lambda D_j$, $x_i \in \Gamma_j$

$$S_i = \partial B\left(x_i, \frac{|x_i-R_j|}{\varsigma}\right), \quad \varsigma > 1$$

$$\nu_i(dx) = \left(\frac{4\pi|x_i-R_j|^2}{\varsigma^2}\right)^{-1} \delta\left(|x-x_i| - \frac{|x_i-R_j|}{\varsigma}\right) dx$$

$$\mathrm{supp}\, \nu_i \subset S_i = \partial B\left(x_i, \frac{|x_i-R_j|}{\varsigma}\right)$$

$$\therefore D(\nu_i, \nu_i) = \frac{1}{2}\left(\frac{|x_i-R_j|}{\varsigma}\right)^{-1} = \frac{\varsigma}{2|x_i-R_j|}.$$

Both cases,

$$\nu_i > 0, \quad \int_{S_i} \frac{\nu_i(dx)}{|x-y|} \le \frac{1}{\underset{x_i-y}{|y-x_i|}} \quad (\forall y \in \mathbb{R}^3)$$

$$2D(\nu_i, \nu_j) = \iint_{S_i \times S_j} \frac{\nu_i(dx)\, \nu_j(dy)}{|x-y|}$$

$$\le \int_{S_j} \frac{\nu_j(dy)}{|x_i - y|} \le \frac{1}{|x_i - x_j|}$$

$$\therefore \sum_{i<j} \frac{1}{|x_i-x_j|} \ge D\Big(\underbrace{\sum_i \nu_i}_{\nu}, \underbrace{\sum_j \nu_j}_{\nu}\Big) - \sum_i D(\nu_i, \nu_i)$$

$$D(\nu, \nu) + \sum_{k<l} \frac{Z^2}{|R_k - R_l|}$$

$$\ge \sum_{i=1}^{M} \underbrace{\int_{S_i} \Phi(x)\, \nu_i(dx)} + \frac{Z^2}{8} \sum_{j=1}^{M} \frac{1}{D_j}$$

Case 1. $S_i \subset \Gamma_j$

$$\int_{S_i} \Phi(x)\, \nu_i(dx) = \sum_{k \neq j} \int_{S_i} \frac{Z}{|x - R_k|}\, \nu_i(dx)$$

$$= \sum_{k \neq j} \frac{Z}{|x_i - R_k|} \quad (\because \text{ def. of } \nu_i)$$

Case 2.

$$\int_{S_i} \Phi(x)\, \nu_i(dx) = \sum_{k=1}^{M} \frac{Z}{|x_i - R_k|} - Z \int_{S_i} \frac{\nu_i(dx)}{D(x)}$$

$$D(x) = \min_{k=1,\dots,M} |x - R_k| = |x - R_j|$$

$x \in S_i$

$$|x - R_k| \geq |x_i - R_k| - \underbrace{|x - x_i|}_{= \frac{|x_i - R_j|}{\varsigma}}$$

$$\geq |x_i - R_j| \left(1 - \frac{1}{\varsigma}\right)$$

$$\therefore D(x) \geq |x_i - R_j| \left(1 - \frac{1}{\varsigma}\right).$$

$$\int_{S_i} \Phi(x)\,\nu_i(dx)$$

$$\geq \sum_{k\neq j} \frac{Z}{|x_i - R_k|} + \frac{Z}{|x_i - R_j|} - \frac{Z}{|x_i - R_j|}\left(1-\frac{1}{\varsigma}\right)^{-1} \underbrace{\int_{S_i} \nu_i(dx)}_{=1}$$

$$1 - \frac{\varsigma}{\varsigma - 1} = -\frac{1}{\varsigma - 1}$$

$$= \sum_{k\neq j} \frac{Z}{|x_i - R_k|} - \frac{1}{\varsigma-1}\cdot\frac{Z}{|x_i - R_j|}.$$

$$\therefore\ V_C(X, R) \geq -\sum_{i=1}^{N} \frac{Z}{D(x_i)} + \frac{Z^2}{8}\sum_{j=1}^{M}\frac{1}{D_j} - \sum_{i=1}^{N} F^\lambda(x_i)$$

Case 1.

$$F^\lambda(x) = \frac{1}{2} D_j^{-1}\left(1 - \frac{|x-R_j|^2}{D_j^2}\right)^{-1}$$

Case 2.

$$F^\lambda(x) = \frac{1}{|x-R_j|}\left(\frac{\varsigma}{2} + \frac{Z}{\varsigma-1}\right) \quad \text{with } \varsigma = 1+\sqrt{2Z} \qquad //$$

Rem. $$(\ \cdots\ ) = \frac{\varsigma-1}{2} + \frac{Z}{\varsigma-1} + \frac{1}{2}$$

$$\geq 2\sqrt{\frac{\cancel{\varsigma-1}}{2}\cdot\frac{Z}{\cancel{\varsigma-1}}} + \frac{1}{2} = \sqrt{2Z} + \frac{1}{2}$$

$$\frac{\varsigma-1}{2} = \frac{Z}{\varsigma-1} \Rightarrow (\varsigma-1)^2 = 2Z.$$

$$\Phi(x) = \int_{\mathbb{R}^3} \frac{\nu(dy)}{|x-y|}$$

$$-\Delta \Phi = 4\pi\nu \quad \text{in } \mathcal{D}'$$

$\forall f \in C_c^\infty(\mathbb{R}^3)$

$$-\int_{\mathbb{R}^3} \Phi(x)\Delta f(x)\,dx = 4\pi \int_{\mathbb{R}^3} f(x)\,\nu(dx)$$

$$\int_{\mathbb{R}^3} \Phi(x)\Delta f(x)\,dx$$

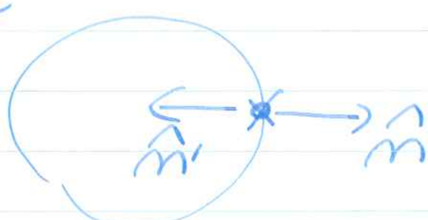
$$\Phi\,\nabla\cdot(\nabla f)$$

$$= \underbrace{\int_{\mathbb{R}^3} \nabla\cdot(\Phi\nabla f)\,dx}_{} - \int_{\mathbb{R}^3} (\nabla\Phi)\cdot(\nabla f)\,dx$$

$$\int_B \nabla\cdot(\Phi\nabla f)\,dx + \int_{B^c} \nabla\cdot(\Phi\nabla f)\,dx$$

$B = B(R_j, D_j)$, $\mathbb{R}^3 = B \cup B^c$

$$= \int_{\partial B} \underset{\substack{\cap \\ C}}{\underline{\Phi}}\ \underset{\substack{\cap \\ C}}{\underline{\nabla f}}\cdot\hat{n}\,dS + \int_{\partial B^c} \underline{\Phi}\ \underline{\nabla f}\cdot\hat{n}\,dS = 0$$



$$= -\int_{\mathbb{R}^3} (\nabla\Phi)\cdot(\nabla f)\,dx$$

$$= -\underbrace{\int_{\mathbb{R}^3} \nabla\cdot(f\nabla\Phi)\,dx} + \underbrace{\int_{\mathbb{R}^3} f\cancel{(x)}\,\Delta\Phi\,dx}$$

$$\underbrace{\int_B f\Delta\Phi\,dx}_{?} + \int_{B^c} f\underbrace{\Delta\Phi}_{=0}\,dx$$

$$-\int_B \nabla\cdot(f\nabla\Phi)\,dx - \int_{B^c} \nabla\cdot(f\nabla\Phi)\,dx$$

$$= -\underbrace{\int_{\partial B} f\nabla\Phi\cdot\hat{n}\,\cancel{dx}\,dS}_{?} - \int_{\partial B^c} f\,\underbrace{\nabla\Phi}_{=\nabla\frac{1}{|x-x_i|}}\cdot\hat{n}\,\cancel{dx}\,dS$$

$$\nabla\frac{1}{|x-x_i|} = -\frac{x-x_i}{|x-x_i|^3}$$

$$= -\int_{\partial B} f\nabla\Phi\cdot\hat{n}\,dS + \int_{\partial B^c} f\,\frac{(x-x_i)\cdot(x-R_j)}{|x-x_i|^3\,|x-R_j|}\,dS$$

$$\hat{n} = \frac{x-R_j}{|x-R_j|} \in S^2$$

$$(x-x_i)\cdot(x-R_j) > 0$$

$\because)$ $(x-R_j+R_j-x_i)\cdot(x-R_j)$

$$= |x-R_j|^2 - (x_i-R_j)\cdot(x-R_j)$$

$$\geq D_j^2 - \lambda D_j\cdot D_j$$

$$= (1-\lambda)D_j^2 > 0$$

$$\int_{\partial B^c} f(x)\, \frac{(x-x_i)\cdot(x-R_j)}{|x-x_i|^3\, D_j}\, dS.$$

$$\int_{\mathbb{R}^3} f\,\nu(dx) = -\frac{1}{4\pi}\int f \,\Delta \Phi\, dx$$

$$= -\frac{1}{4\pi D_j}\int_{\partial B^c} f(x)\, \frac{(x-x_i)\cdot(x-R_j)}{|x-x_i|^3}\, dS$$

$$= \int f(x)\,\nu(dx)$$

$$\nu(dx) = -\underbrace{\frac{(x-x_i)\cdot(x-R_j)}{4\pi D_j\,|x-x_i|^3}}_{=\,\rho(x)}\,\delta(|x-R_j|-D_j)\,dx$$

$$\rho(x) \le \frac{1}{4\pi|x-x_i|^2}$$

$$\Delta \int_{\mathbb{R}^3} \frac{\nu(dy)}{|x-y|}$$

$\Phi(x)$ superharmonic

$\Longrightarrow \nu > 0.$

$$= \int_{\mathbb{R}^3} \nabla \cdot \left(\nabla \frac{1}{|x-y|} \right) \nu(dy)$$

$$= \int_{\mathbb{R}^3} \nabla \cdot \frac{x-y}{|x-y|^3} \nu(dy)$$

$$= \int_{B} \nabla \cdot \frac{x-y}{|x-y|^3} \nu(dy) \quad \operatorname{supp} \nu \subset \Omega_i \subset B$$

$$= \int_{\partial B} \frac{x-y}{|x-y|^3} \cdot \frac{x-R_j}{|x-R_j|} \nu(dy) \quad \geq 0$$

$$(x-y) \cdot (x-R_j) > 0$$

$$\therefore \quad \nu > 0$$