Chapter 5 Electrostatic Inequalities

§5.1 General Properties of the Coulomb Potential

μ : charge distribution
(Borel measure)

$$\Phi(x) = \int_{\mathbb{R}^3} \frac{\mu(dy)}{|x-y|}$$

$\mu > 0 \longrightarrow$ ok

$\mu = \sigma - \tau$
$\sigma, \tau > 0$

$\nu = \sigma + \tau > 0$

$$\int_{\mathbb{R}^3} \frac{\nu(dx)}{1+|x|} < \infty$$

$\Longrightarrow \quad \Phi(x) < \infty$ a.e. $x \in \mathbb{R}^3$

$\Phi \in L^1_{loc}(\mathbb{R}^3)$

$\longrightarrow$ Analysis by Lieb-Loss

Newton's theorem
for spherical symmetric case

Coulomb energy

$$D(\mu,\mu) = \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\mu(dx)\,\mu(dy)}{|x-y|}$$

$\mu > 0 \rightarrow$ ok

$\mu = \sigma - \tau,$

$\rightarrow$ above condition & $D(\sigma,\sigma), D(\tau,\tau) < \infty$

$$\because)\ D(\mu,\mu) = D(\sigma-\tau, \sigma-\tau)$$
$$= D(\sigma,\sigma) + D(\tau,\tau) + 2D(\sigma,\tau)$$

$$2D(\sigma,\tau) \leq D(\sigma,\sigma) + D(\tau,\tau)\ //$$

$\mu(dx) = \rho(x)\,dx$

$$\Rightarrow \underline{D(\rho,\rho)} = \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}\,dx\,dy$$

interaction energy

$$D(\mu,\nu) = \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\mu(dx)\,\nu(dx)}{|x-y|}$$

Prop.

$$\int_{\mathbb{R}^3} \frac{|x-y|}{|x-z|^2|y-z|^2}\,dz = \pi^3$$

$$\forall x,\ y \in \mathbb{R}^3$$

pf. $\begin{cases} y - z = \tilde{z} \\ x - y = \tilde{x} \end{cases}$

$$\Rightarrow I = \int_{\mathbb{R}^3} \frac{|x|}{|x-z|^2\,|z|^2}\,dz$$

$$z = r\omega \in [0,\infty)\times S^2$$

$$I = \int_{S^2} d\omega \int_0^\infty \frac{|x|}{|x-r\omega|^2 \cancel{r^2}} \cdot \cancel{r^2}\,dr$$

$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\quad \omega = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

$$I = \int_0^\infty dr \int_0^\pi \sin\theta\,d\theta\ \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\ \frac{1}{1+r^2-2r\cos\theta}$$

$$\cos\theta = -s \quad \Rightarrow \quad \sin\theta\,d\theta = ds$$

$$\int_{-1}^{1} \frac{ds}{1+r^2+2rs} = \frac{1}{2r}\Big[\log(1+r^2+2rs)\Big]_{-1}^{1}$$

$$= \frac{1}{r}\log\left|\frac{1+r}{1-r}\right|$$

$$I = 2\pi \int_0^\infty \frac{1}{r} \log \frac{1+r}{|1-r|} \, dr$$

$$\frac{I}{2\pi} = \int_0^1 \frac{1}{r} \log \frac{1+r}{1-r} \, dr + \int_1^\infty \frac{1}{r} \log \frac{r+1}{r-1} \, dr$$

$$\begin{aligned} &\log(1+r) = r - \frac{r^2}{2} + \frac{r^3}{3} - \frac{r^4}{4} + \cdots \\ -)\ &\log(1-r) = -r - \frac{r^2}{2} - \frac{r^3}{3} - \frac{r^4}{4} - \cdots \\ \hline &\qquad\qquad = 2\left(r + \frac{r^3}{3} + \frac{r^5}{5} + \cdots\right) \end{aligned}$$

$$\frac{1}{r} \log \frac{1+r}{1-r} = 2\left(1 + \frac{1}{3} r^2 + \frac{1}{5} r^4 + \cdots\right)$$

$$= 2 \sum_{n:\mathrm{odd}} \frac{1}{n} r^{n-1}$$

$$\int_0^1 \frac{1}{r} \log \frac{1+r}{1-r} \, dr = 2 \sum_n{}' \frac{1}{n} \underbrace{\int_0^1 r^{n-1} \, dr}_{= \frac{1}{n}}$$

$$= 2 \sum_n{}' \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n:\mathrm{even}} \frac{1}{n^2} + \sum_{n:\mathrm{odd}} \frac{1}{n^2} = \frac{1}{4} \cdot \frac{\pi^2}{6} + \sum_n{}' \frac{1}{n^2}$$

$$\sum_n{}' \frac{1}{n^2} = \frac{\cancel{3}}{4} \cdot \frac{\pi^2}{\cancel{6}_{\,2}} = \frac{\pi^2}{8}$$

$$\therefore \int_0^1 \frac{1}{r} \log \frac{1+r}{1-r} \, dr = \frac{\pi^2}{4}$$

$$\frac{1}{r} = t \qquad -\frac{dr}{r^2} = dt$$

$$\int_1^{\infty} \frac{1}{r} \log \frac{r+1}{r-1} \, dr$$

$$= -\int_1^0 t \log \frac{\frac{1}{t}+1}{\frac{1}{t}-1} \cdot \left(-\frac{dt}{t^2}\right)$$

$$= \int_0^1 \frac{1}{t} \log \frac{1+t}{1-t} \, dt$$

$$\therefore \int_0^{\infty} \frac{1}{r} \log \left| \frac{1+r}{1-r} \right| dr = 2 \cdot \frac{\pi^2}{4} = \frac{\pi^2}{2} \quad //$$

Theorem 5.1
(Positive Definiteness of the Coulomb Potential)

$$0 \leq D(\mu,\mu) < \infty$$

$$D(\mu,\nu)^2 \leq D(\mu,\mu)\, D(\nu,\nu)$$

pf.

$$\frac{1}{|x-y|} = \frac{1}{\pi^3}\int_{\mathbb{R}^3} \frac{1}{|x-z|^2}\frac{1}{|y-z|^2}\, dz$$

$$D(\mu,\mu) = \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{1}{\pi^3}\int_{\mathbb{R}^3} \frac{1}{|x-z|^2}\frac{1}{|y-z|^2}\, dz \cdot \mu(dx)\mu(dy)$$

$$= \frac{1}{2\pi^3}\int_{\mathbb{R}^3}\left[\int_{\mathbb{R}^3}\frac{\mu(dx)}{|x-z|^2}\right]^2 dz \geq 0$$

$$D(\mu,\mu) = D(\sigma-\tau, \sigma-\tau)$$

$$\leq 2D(\sigma,\sigma) + 2D(\tau,\tau) < \infty$$

$$D(\mu,\nu) = \frac{1}{2\pi^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\mu(dx)}{|x-z|^2}\cdot\int_{\mathbb{R}^3}\frac{\nu(dy)}{|y-z|^2}\, dz$$

$$D(\mu,\nu)^2 \leq \frac{1}{2\pi^3}\int_{\mathbb{R}^3}\left[\int_{\mathbb{R}^3}\frac{\mu(dx)}{|x-z|^2}\right]^2 dz \cdot \frac{1}{2\pi^3}\int_{\mathbb{R}^3}\left[\int_{\mathbb{R}^3}\frac{\nu(dy)}{|y-z|^2}\right]^2 dz$$

$$= D(\mu,\mu)\cdot D(\nu,\nu) \quad //$$

Theorem 5.2
(Newton's Theorem)

$$\mu(RA) = \mu(A)$$

$$\Rightarrow \Phi(x) = \frac{1}{|x|}\int_{|y|\le|x|}\mu(dy) + \int_{|y|>|x|}\frac{1}{|y|}\mu(dy)$$

$$\mu(\{y\in\mathbb{R}^3 : |y|>R\}) = 0$$

$$\Rightarrow \Phi(x) = \frac{1}{|x|}\mu(\mathbb{R}^3) \qquad |x|>R$$

$$\mu(\{y\in\mathbb{R}^3 : |y|<R\}) = 0$$

$$\Rightarrow \Phi(x) = \int_{|y|>R}\frac{1}{|y|}\mu(dy) = \text{const.}$$

$$|x|<R$$

pf.

$$\mu(RA) = \mu(A)$$

$$x, y \in \mathbb{R}^3$$

$$|x| = |y| \implies \Phi(x) = \Phi(y)$$

$$\Phi(x) = \langle \Phi \rangle (x)$$

$$= \frac{1}{4\pi} \int_{S^2} \Phi(|x|\omega)\, d\omega$$

$$\Phi(|x|\omega) = \int_{\mathbb{R}^3} \frac{\mu(dy)}{||x|\omega - y|}$$

$$= \frac{1}{4\pi} \int_{\mathbb{R}^3} \mu(dy) \int_{S^2} \frac{d\omega}{||x|\omega - y|}$$

$$y = \begin{pmatrix} 0 \\ 0 \\ |y| \end{pmatrix} \qquad \omega = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

$$= \frac{1}{4\pi} \int_0^{\pi} d\theta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \frac{\sin\theta\, d\theta}{\sqrt{|x|^2 + |y|^2 - 2|x|\cdot|y|\cos\theta}}$$

$$\cos\theta = -s \implies \sin\theta\, d\theta = ds$$

$$= \frac{2\pi}{4\pi} \int_{-1}^{1} \frac{ds}{\sqrt{|x|^2 + |y|^2 + 2|x|\cdot|y|s}}$$

$$= \frac{1}{2} \cdot \frac{1}{2|x|\cdot|y|} \left[2\sqrt{|x|^2+|y|^2+2|x|\cdot|y|\,s} \,\right]_{-1}^{1}$$

$$= \frac{8}{{}_2\cancel{4}|x|\cdot|y|} \left\{ (|x|+|y|) - \big||x|-|y|\big| \right\}$$

$$= \frac{1}{2|x|\cdot|y|} \begin{cases} 2|y| & |x|>|y| \\ 2|x| & |x|<|y| \end{cases}$$

$$= \begin{cases} \frac{1}{|x|} \\ \frac{1}{|y|} \end{cases} = \min\left\{ \frac{1}{|x|}, \frac{1}{|y|} \right\}$$

$$\Phi(x) = \int_{\mathbb{R}^3} \min\left\{ \frac{1}{|x|}, \frac{1}{|y|} \right\} \mu(dy)$$

$$= \frac{1}{|x|} \int_{|y|\le|x|} \mu(dy) + \int_{|y|>|x|} \frac{\mu(dy)}{|y|} \quad //$$