3.2 Many-Body Hamiltonians

§3.2.1 Many-Body Hamiltonians and Stability: Models with Static Nuclei

Coulomb Hamiltonian

$\alpha := \frac{e^2}{\hbar c} \sim \frac{1}{137}$ Sommerfeld's finestructure constant

$$H_{N,M} = -\frac{1}{2}\sum_{i=1}^{N} \Delta_i + \alpha V_C(\underline{X}, \underline{R})$$

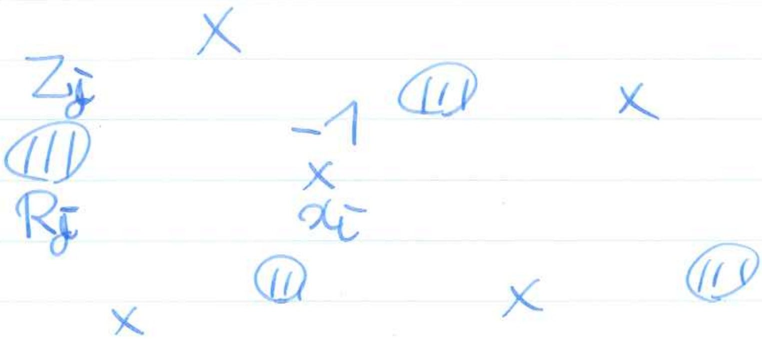
N non-relativistic electrons

M static nuclei fixed at $R_1, \dots, R_M \in \mathbb{R}^3$

$$V_C(\underline{X}, \underline{R}) = W(\underline{X}, \underline{R}) + I(\underline{X}) + U(\underline{R})$$

$$\begin{cases} W(\underline{X}, \underline{R}) = -\sum_{i=1}^{N}\sum_{j=1}^{M} \frac{Z_j}{|x_i - R_j|} \\ I(\underline{X}) = \sum_{1 \le i < j \le N} \frac{1}{|x_i - x_j|} \\ U(\underline{R}) = \sum_{1 \le i < j \le N} \frac{Z_i Z_j}{|R_i - R_j|} \end{cases}$$

$$Z_1, \dots, Z_M \in [0, \infty) \subset \mathbb{R}$$



H : symmetric under permutation of $\{x_i\}$

$\psi \in \bigwedge_{i=1}^{N} L^2(\mathbb{R}^3; \mathbb{C}^q)$ antisymmetric

$q = 2$ for electrons

$$\mathcal{E}_N(\psi) = (\psi, H_{N,M}\psi)$$
$$= \sum_{i=1}^{N} T_\psi^i + \alpha V_\psi$$
$$T_\psi^i = \frac{1}{2} \sum_{\sigma=1}^{q} \int_{\mathbb{R}^{3N}} |(\nabla_{x_i}\psi)(\underline{X},\sigma)|^2 \, dX$$
$$V_\psi = \sum_{\sigma=1}^{q} \int_{\mathbb{R}^{3N}} V_c(\underline{X},\underline{R}) |\psi(X,\sigma)|^2 \, d\underline{X}$$

ground state energy

$$E_N(\underline{Z},\underline{R}) := \inf_{\substack{\psi \in H^1(\mathbb{R}^{3N}) \\ \|\psi\|_2 = 1}} \{ \mathcal{E}_N(\psi) : \psi \text{ bosonic or fermionic} \}$$

$$\begin{cases} \underline{Z} = (Z_1, \dots, Z_M) \in [0,\infty)^M \\ \underline{R} = (R_1, \dots, R_M) \in \mathbb{R}^{3M} \end{cases} \quad \text{fixed}$$

U is unimportant for $E_N(\underline{Z},\underline{R})$

$$V_\psi = -\sum_{i=1}^{N} \sum_{j=1}^{M} Z_j \int \frac{|\psi(\underline{z})|^2}{|x_i - R_j|} \, d\underline{z}$$
$$+ \sum_{1 \le i < j \le N} \int \frac{|\psi(\underline{z})|^2}{|x_i - x_j|} \, d\underline{z}$$
$$+ \sum_{1 \le i < j \le N} Z_i Z_j \boxed{\int \frac{|\psi(\underline{z})|^2}{|R_i - R_j|} \, d\underline{z}}$$

$\frac{1}{|R_i - R_j|}$

Stability of the <u>first</u> kind

$\underset{\text{def}}{\Longleftrightarrow}$ $E_N(\underline{Z}, \underline{R}) > -\infty$ not very hard. See §2.2.1

$$\left(\begin{array}{l} E_j = E_N((0, \dots, Z_j, \dots, 0), R), \quad E_j > -\infty \\ E_N \geq \sum_{j=1}^{M} E_j > -\infty \end{array} \right.$$

<u>absolute</u> ground state energy

$$E_{N,M}(\underline{Z}) := \inf_{R \in \mathbb{R}^{3M}} \{ E_N(\underline{Z}, \underline{R}) \}$$

Stability of the <u>second</u> kind $\longrightarrow$ only for fermions

$\underset{\text{def}}{\Longleftrightarrow}$ $E_{N,M}(\underline{Z}) \geq -\boxed{H}(Z)\,(N+M)$

$$Z := \max\{Z_1, \dots, Z_M\}$$

U plays a decisive role

$(U(R) = 0)$

Why $N^{1/3}$?

$$R_k = 0 \quad (k = 1, \dots, M)$$

$$\Rightarrow E_{N,M}(\underline{Z}) \geq -CN\left(\sum_k Z_k\right)^2 \sim -C\,NM^2$$

for bosons

$$\sim -C\,N^{1/3}M^2$$

for fermions

bulk matter

$$\begin{cases} (2N)^\gamma > N^\gamma + N^\gamma & (\gamma > 1) \\ < & (0 < \gamma < 1) \end{cases}$$

relativistic analogue

$$H^{rel}_{N,M} = \sum_{i=1}^{N} \left(\sqrt{-\Delta_i + 1} - 1 \right) + \alpha V_C(\underline{X}, \underline{R})$$

$$\mathcal{E}^{rel}_N(\psi) = (\psi, H^{rel}_{N,M}\psi) = \sum_{i=1}^{N} T^i_\psi + \alpha V_\psi$$

where $T^i_\psi = \int_{\mathbb{R}^{3N}} \left(\sqrt{(2\pi k_i)^2 + 1} - 1 \right) |\hat{\psi}(\underline{K})|^2 \, d\underline{K}$

ground state energy

$$E^{rel}_N(\underline{Z}, \underline{R}) = \inf_{\substack{\psi \in H^{1/2}(\mathbb{R}^{3N}) \\ \|\psi\|_2 = 1}} \{ \mathcal{E}^{rel}_N(\psi) : \psi \text{ bosonic or fermionic} \}$$

$\hookrightarrow$ finite (See Chap. 8)

$E^{rel}_1(Z)$ finite if $Z\alpha \leq \frac{2}{\pi}$

$Z_i\alpha \leq \frac{2}{\pi}$ $(i = 1, \dots, M) \Rightarrow$ 1^{st} kind (Lemma 8.3)

$+ \quad \alpha < \boxed{?} \Rightarrow$ 2^{nd} kind extreme relativistic limit $m \to 0$

$$|p| - 1 \leq \sqrt{p^2+1} - 1 \leq |p|$$

Fourier trans.

$$\because \quad \frac{p^2}{\sqrt{p^2+1}+1} \leq \frac{p^2}{|p|} = |p|$$

$$\tilde{\mathcal{E}}_N^{rel}(\psi) - N \leq \mathcal{E}_N^{rel}(\psi) \leq \tilde{\mathcal{E}}_N^{rel}(\psi)$$

where $\tilde{\mathcal{E}}_N^{rel}(\psi) = \sum_{i=1}^{N} (\psi, |p_i|\psi) + \alpha V_\psi$

Scaling $\psi \mapsto \psi_\lambda$

$$\psi_\lambda(x_1, \sigma_1; \dots, x_N, \sigma_N) = \lambda^{3N/2} \psi(\lambda x_1, \sigma_1, \dots, \lambda x_N, \sigma_N)$$

$\underline{R} \mapsto \underline{R}/\lambda$

$$\tilde{\mathcal{E}}_N^{rel}(\psi_\lambda) = \lambda \tilde{\mathcal{E}}_N^{rel}(\psi)$$

$$\tilde{E}_N^{rel}(\underline{Z}) = \inf_{\substack{\psi \in H^{1/2}(\mathbb{R}^{3N}) \\ \|\psi\|_2 = 1 \\ \underline{R} \in \mathbb{R}^{3M}}} \left\{ \tilde{\mathcal{E}}_N^{rel}(\psi) \right\} = \begin{cases} -\infty \\ \text{or} \\ 0 \end{cases}$$

$$\begin{cases} \tilde{\mathcal{E}}_N^{rel}(\psi) < 0 \Rightarrow \tilde{E}_N^{rel}(\underline{Z}) = -\infty \\ \qquad\qquad \geq 0 \Rightarrow \qquad\qquad = 0 \end{cases}$$

$$\tilde{\mathcal{E}}_N^{rel}(\psi) \geq 0 \quad \forall \psi, \ \forall \underline{R}$$

$\iff$ 1st kind $\approx$ 2nd kind

Proof of $\tilde{\mathcal{E}}^{rel}_N(\cancel{\lambda}\psi_\lambda) = \lambda\, \tilde{\mathcal{E}}^{rel}_N(\psi)$

$\because)$ $\psi_\lambda(\underline{X}) = \lambda^{3N/2}\psi(\lambda\underline{X})$

$$\int_{\mathbb{R}^{3N}} |\psi_\lambda(\underline{X})|^2 d\underline{X} = \lambda^{3N}\int_{\mathbb{R}^{3N}} |\psi(\lambda\underline{X})|^2 d\underline{X}$$

$$\lambda\underline{X} = \underline{Y} \Rightarrow \lambda^{3N} d\underline{X} = d\underline{Y}$$

$$= \int_{\mathbb{R}^{3N}} |\psi(\underline{Y})|^2 d\underline{Y} = 1$$

$$\tilde{E}^{rel}_N(\psi_\lambda) = \sum_{i=1}^{N} (\psi_\lambda, |p_i|\psi_\lambda) + \alpha V\psi_\lambda$$

$$(\psi_\lambda, |p_i|\psi_\lambda) = \int_{\mathbb{R}^{3N}} |p_i|\cdot|\psi_\lambda(\underline{X})|^2 d\underline{X}$$

$$= \int |2\pi k_i|\cdot|\hat{\psi}_\lambda(\underline{K})|^2 d\underline{K}$$

$$\hat{\psi}_\lambda(\underline{K}) = \int e^{2\pi i \underline{X}\cdot\underline{K}}\psi_\lambda(\underline{X})\, d\underline{X}$$

$$= \lambda^{3N/2}\int e^{2\pi i K\cdot X}\cdot\psi(\lambda\underline{X})\, d\underline{X}$$

$$\lambda\underline{X} = \underline{Y} \Rightarrow \lambda^{3N} d\underline{X} = d\underline{Y}$$

$$= \lambda^{-3N/2}\int e^{2\pi i \frac{K}{\lambda}\cdot Y}\psi(\underline{Y})\, d\underline{Y}$$

$$= \lambda^{-3N/2}\hat{\psi}\left(\frac{\underline{K}}{\lambda}\right)$$

$$= \lambda\cdot\int \left|2\pi\frac{\underline{K}}{\lambda}\right|\cdot\left|\hat{\psi}\left(\frac{\underline{K}}{\lambda}\right)\right|^2 d\underline{K} \times \lambda^{-3N}$$

$$= \lambda\int |2\pi\underline{K}|\cdot|\hat{\psi}(\underline{K})|^2 d\underline{K}$$

$$= \lambda(\psi, |p_i|\psi)$$

(cont...)

In V_{ψ_λ}

$$\int \frac{|\psi_\lambda(\underline{x})|^2}{|x_i - \frac{R_j}{\lambda}|} d\underline{x} = \lambda^{3N} \cdot \lambda \int \frac{|\psi(\lambda \underline{x})|^2}{|\lambda x_i - R_j|} d\underline{x}$$

$$= \lambda \int \frac{|\psi(\lambda \underline{x})|^2}{|\lambda x_i - R_j|} \cdot \lambda^{3N} d\underline{x}$$

$$= \lambda \int \frac{|\psi(\underline{x})|^2}{|x_i - R_j|} d\underline{x} \quad \text{etc.} \qquad //$$

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$\mathbb{B}(x) = \nabla \times \mathbb{A}(x)$ (See Chap. 10)

for spinless particles

$$\mathbb{p} \to \mathbb{p} + \sqrt{\alpha}\,\mathbb{A}(x)$$

i.e. $-\Delta \mapsto \left(-i\nabla + \sqrt{\alpha}\,\mathbb{A}(x)\right)^2$

$$\begin{cases} H_{N,M} \mapsto H_{N,M}(\mathbb{A}) & \text{diamagnetic ineq. (Chap. 4)} \\ H^{rel}_{N,M} \mapsto H^{rel}_{N,M}(\mathbb{A}) & \text{much more complicated} \\ \quad \text{not significantly affect} & \text{(Chap. 8)} \end{cases}$$

$$E_{N,M}(\underline{Z}) = \inf_{\mathbb{B},\underline{R}} \{ E_N(\underline{Z}, \underline{R}, \mathbb{B}) \}$$
$$\geq -\Xi(Z)(N+M) \quad \text{for fermions}$$

fermions with spin

$$H_{N,M} \mapsto H_{N,M}(\mathbb{A}) - \frac{\sqrt{\alpha}}{4} g \sum_{i=1}^{N} \sigma_i \cdot \mathbb{B}(x_i)$$

$g = 2$ gyromagnetic ratio (2.002319)

$\sigma = (\sigma^1, \sigma^2, \sigma^3)$ Pauli

$\mathcal{E}_N(\psi) \to -\infty$ if "$\mathbb{B} \to \infty$"

$$\mathcal{E}_{mag}(\mathbb{B}) = \frac{1}{8\pi} \int_{\mathbb{R}^3} |\mathbb{B}(x)|^2\, dx \quad \text{magnetic field energy}$$

$$\left(\begin{array}{l} Z\alpha^2,\ \alpha \text{ not too large} \\ 0 \leq g \leq 2 \end{array} \right. \Rightarrow \text{stability}$$

§3.2.2 Many-Body Hamiltonians : Models without Static Particles

Bosons with static nuclei (non-relativistic)

$$E_{N,M}(\underline{Z}) \text{ grows as } -(\min\{N,M\})^{\frac{5}{3}}$$

Bosons without static nuclei

$$E_{N,M}(\underline{Z}) \text{ grows as } -(\min\{N,M\})^{\frac{7}{5}}$$

$\longrightarrow$ Chap. 7

Dynamic nuclei

$$\mathcal{E}_N(\psi) \longmapsto \mathcal{E}_N(\psi) + \sum_{j=1}^{M} T_\psi^j$$

$$T_\psi^j = \frac{1}{2\mu}\int |\nabla_{R_j}\psi|^2 d\underline{R}$$

$$\underset{\displaystyle L^2(\mathbb{R}^{3(N+M)})}{\underset{\cap}{\psi}} = \psi(\underline{X},\underline{R}) = \psi(x_1,\dots,x_N;R_1,\dots,R_M)$$

Relativistic gravitating system ($\to$ Chap. 13)

e.g. neutrons

$6.674 \times 10^{-8}\ g^{-1} cm^3 sec^{-2}$

$$\text{Hamiltonian} = \sum_{i=1}^{N} \left(\sqrt{-\Delta_i + m_m^2} - m_m \right) - G m_m^2 \sum_{1 \le i < j \le N} \frac{1}{|x_i - x_j|}$$

$$\sum_{i=1}^{N} \left(\sqrt{-\Delta_i + 1} - 1 \right) - K \sum_{i,j} \frac{1}{|x_i - x_j|}$$

$$K = \frac{G m_m^2}{\hbar c} \simeq 7 \times 10^{\cancel{-37}\,-39}$$

$$\mathcal{E}_N^{grav}(\psi) = \sum_{i=1}^{N} T_\psi^i - K \left(\psi, \sum_{i,j} \frac{1}{|x_i - x_j|} \psi \right)$$

$$T_\psi^i = \int \left(\sqrt{(2\pi k_i) + 1} - 1 \right) |\hat{\psi}(\underline{k})|^2 \, d\underline{k}$$

$N \ge N_c$, Chandrasekhar mas limit

$\Rightarrow \mathcal{E}_N^{grav}(\psi) \to -\infty$ (See Chap. 13)

White dwarfs x (|||) He x

(|||) x (|||)

$$\sum_{i=1}^{N} T_\psi^{i,e} + \sum_{j=1}^{M} T_\psi^{j,m}$$

$$-(\alpha Z_1 + K m_m^{-1}) \left(\psi, \sum_i \sum_j \frac{1}{|x_i - R_j|} \psi \right)$$

$$+ (\alpha Z^2 - K) \left(\psi, \sum_{i,j} \frac{1}{|R_i - R_j|} \psi \right)$$

$$+ (\cancel{\alpha} Z_1 - K m_m^{-2}) \left(\psi, \sum_{i,j} \frac{1}{|x_i - x_j|} \psi \right)$$

Proof.

Z_k appears <u>linearly</u> in $H_{N,M}$ for Z_j fixed ($\forall j \neq k$)

$$H_{N,M} = -\frac{1}{2}\sum_{i=1}^{N}\Delta_i + \alpha V_c(\underline{X}, \underline{R})$$

$$= -\frac{1}{2}\sum_i \Delta_i$$

$$\underbrace{-\alpha \sum_i \sum_j \frac{Z_j}{|x_i - R_j|}} + \alpha \sum_{i,j} \frac{1}{|x_i - x_j|} + \alpha \boxed{\sum_{i,j} \frac{Z_i Z_j}{|R_i - R_j|}}$$

$$\left\{ \begin{array}{l} \sum_i \frac{1}{|x_i - R_k|} \cdot Z_k + \sum_{j \neq k} \sum_i \frac{Z_j}{|x_i - R_j|} \\ \sum_{j \neq k} \frac{Z_j}{|R_i - R_j|} \cdot Z_k + \sum_{\substack{i<j \\ j, i \neq k}} \frac{Z_i Z_j}{|R_i - R_j|} \end{array} \right.$$

$$= H^{(1)} \cdot \underline{Z} + H^{(2)} \longrightarrow \text{affine (operator-valued) function}$$

where $H_k^{(1)} = -\alpha \sum_i \frac{1}{|x_i - R_k|} + \alpha \sum_{j \neq k} \frac{Z_j}{|R_i - R_j|}$

$$H_k^{(2)} = -\frac{1}{2}\sum_i \Delta_i - \alpha \sum_{j \neq k} \sum_i \frac{Z_j}{|x_i - R_j|} + \alpha \sum_{i,j} \frac{1}{|x_i - x_j|} + \sum_{\substack{i<j \\ i,j \neq k}} \frac{Z_i Z_j}{|R_i - R_j|}$$

$$(\psi, H_{N,M}\psi) = (\psi, H^{(1)} \cdot \underline{Z}\psi) + (\psi, H^{(2)}\psi)$$

$$Z_k = (1-\lambda) \cdot 0 + \lambda Z \qquad \lambda \in [0, 1]$$

$$\inf_\psi (\psi, H_{N,M}\psi)$$

$$\geq (1-\lambda) \inf_\psi (\psi, H_{N,M}\psi)\Big|_{Z_k = 0} + \lambda \inf_\psi (\psi, H_{N,M}\psi)\Big|_{Z_k = Z}$$

$$\therefore E_N(\underline{Z}, \underline{R}) \geq (1-\lambda) E_N((z_1, \dots, \overset{k}{0}, \dots, z_M), \underline{R})$$

$$+ \lambda E_N((z_1, \dots, Z, \dots, z_M), \underline{R})$$

concave!

$\forall z_k \in [0, Z]$

$$\Rightarrow E_N(\underline{Z}, \underline{R}) \geq E_N(\underline{Z}, \underline{R})\Big|_{z_k = 0 \text{ or } z_k = Z}$$

$$\because) \quad f(\underbrace{x}_{(1-\lambda)a + \lambda b}) \geq (1-\lambda) f(a) + \lambda f(b)$$

$$\exists x_0 \begin{cases} f(x_0) < f(a) \\ f(x_0) < f(b) \end{cases}$$

$$\Rightarrow f(\underbrace{x_0}_{\exists \lambda \ (1-\lambda)a + \lambda b}) < (1-\lambda) f(a) + \lambda f(b)$$

contradiction //

$\forall k = 1, \dots, M. \quad \Rightarrow$ completed.

§3.2.3 Monotonicity in the Nuclear Charges

Prop. $Z_k \leq Z \quad (k=1,\dots,M)$

$$\Rightarrow E_N(\underline{Z}, \underline{R}) \geq \min_{\underline{\tilde{R}} \subset \underline{R}} E_N((Z,\dots,Z), \tilde{R})$$

Moreover, $Z_k \leq \tilde{Z}_k \quad (k=1,\dots,M)$

$$\Rightarrow E_{N,M}(\underline{Z}) \geq E_{N,M}(\underline{\tilde{Z}}).$$

$$\sum_{i,j} \frac{Z_i Z_j}{|R_i - R_j|} \mapsto \sum \frac{Z_i Z_j}{R|R_i - R_j|} = \frac{\frac{Z_i}{\sqrt{R}} \frac{Z_j}{\sqrt{R}}}{|R_i - R_j|} \qquad \to 0$$

$$R_i \Leftarrow R R_i \qquad R \to \infty$$

$$\sum_{i,j} \frac{Z_j}{|x_i - R_j|} \mapsto \sum \frac{Z_j}{R\left|\frac{x_i}{R} - R_j\right|}$$

$$R \to \infty \Rightarrow \frac{Z_j}{R} \to 0, \quad \frac{x_i}{R} \to 0$$

$$\sum \frac{1}{|x_i - x_j|} \mapsto \sum \frac{1}{R\left|\frac{x_i}{R} - \frac{x_j}{R}\right|}$$

$E_{N,M}(\underline{Z})$ not increasing for small Z_k

$Z_k = 0 \iff |R_k| \to \infty$

$$E_{N,M}(\underline{Z}) \leq E_{N,M}(\underline{Z})\big|_{Z_k = 0}$$

$$\therefore \min_{Z_k} E_{N,M}(\underline{Z}) = E_{N,M}(\underline{Z})\big|_{Z_k = \tilde{Z}_k}$$

monotone decrease

Theorem 3.3
(Symmetry of Minimizers)

$$\mathcal{E}(\psi)$$

$$\begin{array}{cccc} \mathcal{E} : & L^2(\mathbb{R}^{dN}) & \longrightarrow & \mathbb{C} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & \psi & \longmapsto & \mathcal{E}(\psi) \end{array}$$

quadratic form

(a) $\mathcal{E}(\psi) \geq {}^{\exists}C(\psi, \psi)$

(b) $\mathcal{E}(\psi) \geq \mathcal{E}(|\psi|)$

(c) $\mathcal{E}(\psi_\pi) = \mathcal{E}(\psi) \quad \forall \pi \in S_N$

$$\psi_\pi(x_1, \dots, x_N) = \psi(x_{\pi(1)}, \dots, x_{\pi(N)})$$

$$\Longmapsto \quad E_0 := \inf_{(\psi,\psi)=1} \{ \mathcal{E}(\psi) \}$$

$$= \inf_{(\psi,\psi)=1} \{ \mathcal{E}(\psi) : \psi_\pi = \psi \ \ \forall \pi \in S_N \}$$

Proof.

(a) $\Longrightarrow$ $\mathcal{E}(\psi) \geq c(\psi, \psi) = c$

$$E_0 = \inf_{(\psi,\psi)=1} \{\mathcal{E}(\psi)\} \geq c \quad (\text{finite})$$

WLOG $\psi = |\psi|$

$\because$) (b) $\Longrightarrow$ $\mathcal{E}(\psi) \geq \mathcal{E}(|\psi|)$

$(|\psi|, |\psi|) = (\psi, \psi)$

$\because$) $\text{LHS} = \int \underbrace{|\overline{\psi(\underline{x})}| \cdot |\psi(\underline{x})|}_{\substack{\| \\ |\psi(\underline{x})|^2 \\ \| \\ \overline{\psi(\underline{x})}\,\psi(x)}} d\underline{x}$

$= \text{RHS}$ //

$$\cancel{\mathcal{E}(\psi)}\ E_0 = \inf_{(\psi,\psi)=1} \{\mathcal{E}(\psi)\} \geq \inf_{\underbrace{(\psi,\psi)=1}_{\hookrightarrow (|\psi|, |\psi| = 1)}} \mathcal{E}(|\psi|)$$

$\psi = e^{i\theta}|\psi| \Longrightarrow \theta = 0.$ //

ψ : symmetric $\Longrightarrow$ $|\psi|$: symmetric

$(\psi_\pi = \psi)$

$\psi \in L^2(\mathbb{R}^{dN})$, $\psi \geq 0$.

$\psi = \psi_s + \psi_r$

where $\psi_s = \frac{1}{N!} \sum_{\pi \in S_N} \psi_\pi$

$\psi_\pi \geq 0 \Rightarrow \psi_s \geq 0$

$$\boxed{\mathcal{E}(\psi) = \mathcal{E}(\psi_s) + \mathcal{E}(\psi_r)}$$

$\because)$ $\mathcal{E}(\psi) = (\psi, H\psi)$ $\quad H$ on $L^2(\mathbb{R}^{dN})$ ~~$\to L^2(\mathbb{R}^{dN})$~~

$\cancel{\hat{\mathcal{E}}(\phi, \psi) := (\phi, H\psi)}$

$$\mathcal{E}(\phi \pm \psi) = (\phi \pm \psi, H\phi \pm H\psi)$$

$$= (\phi, H\phi) + (\psi, H\psi)$$

$$\pm (\phi, H\psi) \pm (\psi, H\phi)$$

$$\mathcal{E}(\phi \pm i\psi) = (\phi, H\phi) + (\psi, H\psi)$$

$$\pm i(\phi, H\psi) \mp i(\psi, H\phi)$$

$$\tilde{\mathcal{E}}(\phi, \psi) := \frac{1}{4}\mathcal{E}(\phi+\psi) - \frac{1}{4}\mathcal{E}(\phi-\psi) + \frac{1}{4i}\mathcal{E}(\phi+i\psi) - \frac{1}{4i}\mathcal{E}(\phi-i\psi)$$

$$\left(= (\phi, H\psi) \right)$$

$\cancel{\hat{\mathcal{E}}(\psi_s, \psi_r)}$

$\mathcal{E}(\psi)$

$= \mathcal{E}(\psi_s + \psi_r)$

$= \hat{\mathcal{E}}(\psi_s + \psi_r, \ \psi_s + \psi_r)$

$= \hat{\mathcal{E}}(\psi_s, \psi_s) + \hat{\mathcal{E}}(\psi_r, \psi_r)$

$\quad + \underline{\hat{\mathcal{E}}(\psi_s, \psi_r)} + \hat{\mathcal{E}}(\psi_r, \psi_s)$

$\hat{\mathcal{E}}(\psi_s, \psi_r) \qquad \psi - \psi_s$

$$= \left(\frac{1}{N!} \sum_{\pi} \psi_\pi, \ \psi - \frac{1}{N!} \sum_{\sigma} \psi_\sigma \right)$$

$$\frac{1}{N!} \sum_{\sigma} (\psi - \psi_\sigma)$$

$$= \frac{1}{(N!)^2} \sum_{\pi} \sum_{\sigma} \hat{\mathcal{E}}(\psi_\pi, \ \psi - \psi_\sigma) \qquad = 0.$$

$$\hat{\mathcal{E}}(\psi, \ \psi_{\pi^{-1}} - \psi_{\pi^{-1}\sigma})$$

$$\{\psi_{\pi^{-1}}\} = \{\psi_{\pi^{-1}\sigma}\}$$

$\therefore \ \mathcal{E}(\psi) = \mathcal{E}(\psi_s) + \mathcal{E}(\psi_r)$ //

$H = \mathbb{1} \implies (\psi, \psi) = (\psi_s, \psi_s) + (\psi_r, \psi_r).$

$$\boxed{(\psi_s, \psi_s) \geq \frac{1}{N!}}$$

$$\because) \quad (\psi_s, \psi_s) = \frac{1}{N!^2} \sum_{\pi} \sum_{\pi'} \underbrace{(\psi_\pi, \psi_{\pi'})}_{\cancel{\geq 0}}$$

$$= \frac{1}{N!^2} \left\{ \sum_{\pi = \pi'} \underbrace{(\psi_\pi, \psi_\pi)}_{\substack{\parallel \\ (\psi, \psi) = 1.}} + \sum_{\pi \neq \pi'} \underbrace{(\psi_\pi, \psi_{\pi'})}_{\geq 0} \right\}$$

$$\geq \frac{1}{N!^2} \underbrace{\sum_{\pi \in S_N} 1}_{= N!} = \frac{1}{N!} \quad // \quad > 0.$$

$$E_0 = \varepsilon(\psi)$$

$$= \varepsilon(\psi_s) + \varepsilon(\psi_r)$$

$$\geq (\psi_s, \psi_s) E_0 + (\psi_r, \psi_r) E_0$$

$$= (\psi, \psi) E_0$$

$$= E_0.$$

$$\therefore \quad \varepsilon(\psi_s) = (\psi_s, \psi_s) E_0$$

$$\varepsilon\left(\frac{\psi_s}{\|\psi_s\|}\right) = E_0 \quad //$$

$$\psi^{(n)}, \quad \lim_{n\to\infty} \mathcal{E}(\psi^{(n)}) = E_0$$

$$\psi^{(n)} = \psi_s^{(n)} + \psi_r^{(n)}$$

$$E_0 + \epsilon > \mathcal{E}(\psi^{(n)}) \qquad \forall \epsilon > 0 \quad n \geq {}^{\exists} n_0$$

$$= \mathcal{E}(\psi_s^{(n)}) + \mathcal{E}(\psi_r^{(n)})$$

$$\geq (\psi_s^{(n)}, \psi_s^{(n)}) E_0 + (\psi_r^{(n)}, \psi_r^{(n)}) E_0$$

$$= (\psi^{(n)}, \psi^{(n)}) E_0 = E_0.$$

$$\mathcal{E}(\psi_s^{(n)}) \geq (\psi_s^{(n)}, \psi_s^{(n)}) E_0$$

$$\mathcal{E}\left(\frac{\psi_s^{(n)}}{\|\psi_s^{(n)}\|}\right) \geq E_0$$

$$\forall \epsilon > 0 \ {}^{\exists} n_0 \ \text{s.t.} \ \left| \mathcal{E}(\psi_s^{(n)}) / \|\psi_s^{(n)}\|^2 - E_0 \right| < \epsilon$$

$$\underbrace{E_0 - \epsilon < \mathcal{E}(\psi_s^{(n)})}_{\text{ok}} / \|\psi_s^{(n)}\|^2 < E_0 + \epsilon$$

$$E_0 + \epsilon > \mathcal{E}(\psi_s^{(n)}) + \mathcal{E}(\psi_r^{(n)}) \geq \mathcal{E}(\psi_s^{(n)})$$