$$
\begin{tikzcd}
\lambda \in \tilde{\mathcal{F}}_N \arrow[d, "T_\varepsilon"] \\
f \in \mathcal{F}_N \arrow[d, "\tau"] \\
\tilde{f} \in \tilde{\mathcal{F}}_N \arrow[u, bend left=60, "T_\varepsilon"']
\end{tikzcd}
\qquad
\lambda_j = \varepsilon \chi_{[1,N]}(j) + (1-\varepsilon) f_j
$$

$$\tau : \mathbb{N} \longrightarrow \mathbb{N} \setminus \{N\} \quad \varepsilon = \lambda_N$$

$$\tilde{f}_j := f_{\tau(j)}$$

$$f_j = \begin{cases} \tilde{f}_{\sigma(j)} & j \neq N \\ 0 & j = N \end{cases}$$

$$\varepsilon_2 = \tilde{f}_N$$

$$\tilde{f}_j = \varepsilon_2 \chi_{[1,N]}(j) + (1-\varepsilon_2) f_j^{(2)}$$

$$\lambda_N = \overset{\varepsilon_1}{\cancel{\lambda_N}} \cdot 1 + f_N = \varepsilon_1$$

$$\boxed{j \neq N}$$

$$\chi_2(j) = \begin{cases} 1 & \vdots \\ 0 & j \geq 2 \end{cases}$$

$$f_j = \tilde{f}_{\sigma(j)} = \varepsilon_2 \overbrace{\chi_{[1,N]}(\sigma(j))}^{\chi_2(j)} + (1-\varepsilon_2) f^{(2)}_{\sigma(j)}$$

$$\lambda_j = \sum_{k=1}^{2} \underbrace{\varepsilon_k \prod_{i=1}^{k-1} (1-\varepsilon_k) \cdot \chi_k(j)}_{c_k} + f^{(2)}_{\sigma(j)} \prod_{i=1}^{2} (1-\varepsilon_i)$$

$$f^{(2)} \xrightarrow[\tau_2]{} \widetilde{f}^{(2)} \qquad \widetilde{f}^{(2)}_j := f^{(2)}_{\tau_2(j)}$$

$$\widetilde{f}^{(1)}_N \qquad \tau_2 : \mathbb{N} \to \mathbb{N} \setminus \{N\}$$

~~if $\varepsilon_2 = \widetilde{f}^{(1)}_N$~~

$$f^{(2)}_j = \begin{cases} \widetilde{f}^{(2)}_{\sigma_2(j)} & j \neq N \\ 0 & j = N \end{cases}$$

$$f^{(2)}_{\sigma_1(j)} = \begin{cases} \widetilde{f}^{(2)}_{\sigma_2(\sigma_1(j))} & j \neq N, \ N+1 \\ 0 & j = N+1 \end{cases}$$

$$\lambda_{N+1} = \sum_{k=1}^{2} \varepsilon_k \prod_{i=1}^{k-1} (1-\varepsilon_k)\, x_k(\cancel{j}\ N+1)$$

$$x_1(N+1) = 0$$
$$x_2(N+1) = 1$$

$$= \varepsilon_2 (1-\varepsilon_1)$$
$$1-\lambda_N$$

$$\varepsilon_2 = \frac{\lambda_{N+1}}{1-\varepsilon_1} = f_{N+1} = \widetilde{f}_N$$

$\underline{j \neq N, N+1}$

$$\lambda_j = \sum_{k=1}^{3} \varepsilon_k \prod_{i=1}^{k-1} (1-\varepsilon_k) \cdot x_k(j) + \underbrace{f^{(3)}_{\sigma_2\sigma_1(j)}}_{R^{(3)}_j} \prod_{i=1}^{3} (1-\varepsilon_i)$$

$3 \longrightarrow M$

$j \neq N, N+1, \ldots, N+M-\cancel{1}2$

$R^{(M)}_j$ defined for

$$r^{(M)}_j = \begin{cases} f^{(M)}_{\sigma_{M-1}\cdots\sigma_1(j)} & j < N \\ 0 & j = N+M-\cancel{2}1 \\ \underline{\qquad} & j > N+M-\cancel{2}1 \end{cases}$$

$$f^{(3)}_j = \begin{cases} \tilde{f}^{(3)}_{\sigma_3(j)} & j \neq N \\ 0 & j = N \end{cases}$$

$$f^{(3)}_{\sigma_2\sigma_1(j)} = \begin{cases} \tilde{f}^{(3)}_{\sigma_3\sigma_2\sigma_1(j)} & j = N, N+1, N+2. \\ 0 & j = N+2 \end{cases}$$

$f^{(3)}_N = 0.$

j = fix $\qquad r_j^{(m)} \longrightarrow 0 \quad (\text{as } m \to \infty)$

$$r_j^{(m)} \overset{\tau}{\longmapsto} \underset{\underset{\mathfrak{F}_N}{\in}}{\tilde{r}_j^{(m)}} \qquad \tilde{r}_j^{(m)} = r^{(m)}_{\tau(j)}$$

$$\varepsilon_m = \min\{\tilde{r}_N^{(m)},\ 1-\tilde{r}_{N+1}^{(m)}\}$$

$$1-\tilde{r}_{N+1}^{(m)} \geq \frac{1}{N+1} > 0$$

$$\cancel{R_j^{(m)} = r_{N_j}^{(m)}\left(1-\sum_k c_k\right)}$$

$$\cancel{\begin{array}{ccc} R_j^{(m)} & & \left(1-\sum_k c_k\right) \\ \downarrow & & \downarrow \\ R_j & & (1-d) \\ R_j^{(m)} & & \end{array}}$$

$$R_j^{(m)} \longrightarrow \exists R_j$$

$$r_j^{(m)} = \frac{R_j^{(m)}}{1-\sum_k c_k} \qquad \tilde{R}_j^{(m)} := R^{(m)}_{\tau_m(j)}$$

$$\tau \downarrow$$

$$\underset{\underset{\mathfrak{F}_N}{\in}}{\tilde{r}_j^{(m)}} = \frac{\tilde{R}_j^{(m)}}{1-\sum_k c_k}$$

$R_j^{(m)} \longrightarrow \exists R_j$

$$r_j := \frac{R_j}{1-d}$$

$\tau\ I$

$$\tilde{r}_j = \frac{R_{\tau(j)}}{1-d} \in \tilde{\mathcal{F}}_N$$

$$\varepsilon = \min\{\tilde{r}_N,\ 1-\tilde{r}_{N+1}\}$$

$$\tilde{r}_N > 0$$

$$\tilde{r}_j \xmapsto{T_\varepsilon} r_j' \xmapsto{\tau} \tilde{r}_j$$

$$\varepsilon_m = \min\{\tilde{r}_N^{(m)},\ 1-\tilde{r}_{N+1}^{(m)}\}$$

$$\tilde{r}_N^{(m)} := r^{(m)}_{\tau_m(N)}$$

$$r_j^{(m)} = f^{(m)}_{\sigma_{m-1}\cdots\sigma_1(j)}$$

$r_j^{(m)}$

$\tau\ I$

$\mathcal{F} \to$ $\tilde{\mathcal{F}}$ 中 $\quad \tilde{r}_j^{(m)} := r^{(m)}_{\tau(j)}$

$\downarrow$

$\tilde{r}_j$ $\quad \forall m \geq m_0 \quad \tilde{r}_j^{(m)} > \frac{\tilde{r}_j}{2}$

$$\tilde{r}_N^{(m)} > \frac{\tilde{r}_N}{2} > 0$$

$$R_j^{(m)} \longrightarrow \exists R_j$$

$$r_j := \frac{R_j}{1-\alpha} \in \mathcal{F}$$

$$\tau \downarrow$$

$$\widetilde{r_j} \in \widetilde{\mathcal{F}}$$

$$r_j^{(m)}$$

$$\tau \downarrow$$

$$\widetilde{r}_j^{(m)} := r_{\tau(j)}^{(m)} \notin \widetilde{\mathcal{F}}_N \text{ in general}$$

$$\downarrow m \to \infty$$

$$\widetilde{r_j}$$

$$\forall m \geq m_0 \quad \widetilde{r}_j^{(m)} \geq \frac{\widetilde{r_j}}{2}$$

$$\therefore \widetilde{r}_N^{(m)} \geq \frac{\widetilde{r_N}}{2} > 0$$

$$\varepsilon_m = \min\{\widetilde{r}_N^{(m)}, 1 - \widetilde{r}_{N+1}^{(m)}\}$$

$$\geq \min\left\{\frac{\widetilde{r_N}}{2}, \frac{1}{N+1}\right\} > 0.$$