Example

$$\lambda = \{1, \dots, 1, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\} \in \tilde{\mathcal{F}}$$

$$\lambda_j = \begin{cases} 1 & (j = 1, \dots, N-1) \\ \frac{1}{2^{j-N+1}} & (j = N, N+1, \dots) \end{cases}$$

$$\varepsilon = \min\{\frac{1}{2}, 1 - \frac{1}{4}\} = \frac{1}{2} \ (= \lambda_N)$$

$$T_\varepsilon : \lambda \mapsto f = \{1, \dots, 1, 0, \frac{1}{2}, \frac{1}{2^2}, \dots\} \in \mathcal{F}$$

$$\tau \Downarrow$$

$$\hat{f} = \{1, \dots, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\} \in \tilde{\mathcal{F}}$$

$$\tau = \begin{pmatrix} 1 & \cdots & N-1 & N & N+1 & \cdots \\ 1 & \cdots & N-1 & N+1 & N+2 & \cdots \end{pmatrix}$$

$$\tau : \underset{\mathcal{F}}{\underset{\cap}{f}} \mapsto \underset{\tilde{\mathcal{F}}}{\underset{\cap}{\hat{f}}} \quad \text{s.t. } \hat{f}_j = f_{\tau(j)}$$

injective but not surjective!

$$\sigma = \begin{pmatrix} 1 & \cdots & N-1 & N+1 & N+2 & \cdots \\ 1 & \cdots & N-1 & N & N+1 & \cdots \end{pmatrix}$$

$$f_j = \begin{cases} \tilde{f}_{\sigma(j)} & (j \neq N) \\ 0 & (j = N) \end{cases}$$

$\underline{j \neq N}$

$$\begin{aligned} \lambda_j &= \varepsilon \underbrace{\chi_{[1,N]}(j)} + (1-\varepsilon)\tilde{f}_{\sigma(j)} \\ &= \varepsilon \chi_1(j) \\ &\quad + (1-\varepsilon)\left\{ \varepsilon_2 \underbrace{\chi_{[1,N]}(\sigma(j))}_{\substack{\| \\ \chi_2(j)}} + (1-\varepsilon_2)\tilde{f}^{(2)}_{\sigma(j)} \right\} \\ &= \sum_{k=1}^{2} \varepsilon_k \prod_{i=1}^{k-1}(1-\varepsilon_i)\chi_k(j) + f^{(2)}_{\sigma(j)} \prod_{i=1}^{2}(1-\varepsilon_i) \end{aligned}$$

$\underline{j = N}$

$$\lambda_N = \varepsilon \chi_{[1,N]}(N) = \varepsilon = \frac{1}{2}$$

$f^{(1)} = f, \quad \tilde{f}^{(1)} = \tilde{f}, \quad \varepsilon_1 = \varepsilon, \quad$ etc...

$$\begin{array}{ccccc} \tilde{f}^{(1)} & \xmapsto{T_{\varepsilon_2}} & f^{(2)} & \xmapsto{\tau_2} & \tilde{f}^{(2)} \\ \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\ \tilde{\mathcal{F}} & & \mathcal{F} & & \tilde{\mathcal{F}} \end{array}$$

$$\tilde{f}^{(2)}_j = f^{(2)}_{\tau(j)}$$

$$f^{(2)}_j = \begin{cases} \tilde{f}^{(2)}_{\sigma(j)} & (j \neq N) \\ 0 & (j = N) \end{cases}$$

$$f^{(2)}_{\sigma(j)} = \tilde{f}^{(2)}_{\sigma^2(j)} \quad (j \neq N)$$

$\underline{j \neq N, N+1}$

$$\lambda_j = \sum_{k=1}^{3} \varepsilon_k \prod_{i=1}^{k-1} (1-\varepsilon_i) x_k(j) + f^{(3)}_{\sigma^2(j)} \prod_{i=1}^{3} (1-\varepsilon_i)$$

$\underline{j = N+1}$

$$\lambda_{N+1} = \sum_{k=1}^{2} \varepsilon_k \prod_{i=1}^{k-1} (1-\varepsilon_i) = \frac{1}{2^2}$$

$$\left(\begin{aligned} \because x_2(N+1) &= x_{[1,N]}(\sigma(N+1)) \\ &= x_{[1,N]}(N) = 1 \end{aligned} \right)$$

$$\tilde{f}^{(m-1)} = \{1, \dots, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\} \in \tilde{\mathcal{F}}$$

$$\downarrow T_{\varepsilon_m} = T_\varepsilon$$

$$f^{(m)} = \{1, \dots, 1, 0, \frac{1}{2}, \frac{1}{2^2}, \dots\} \in \mathcal{F}$$

$$\downarrow \tau_m = \tau$$

$$\tilde{f}^{(m)} = \{1, \dots, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\} \in \tilde{\mathcal{F}}$$

$$\tilde{f}^{(m)}_j = f^{(m)}_{\tau(j)}$$

$$f^{(m)}_j = \begin{cases} \tilde{f}^{(m)}_{\sigma(j)} & (j \neq N) \\ 0 & (j = N) \end{cases}$$

$$f^{(m)}_{\sigma^{m-1}(j)} = \tilde{f}^{(m)}_{\sigma^m(j)} \quad j \notin \{N, N+1, \dots, N+m-1\}$$

$$\lambda_j = \sum_{k=1}^{m} \varepsilon_k \prod_{i=1}^{k-1} (1-\varepsilon_i) \cdot \chi_k(j) + f^{(m)}_{\sigma^{m-1}(j)} \prod_{i=1}^{m} (1-\varepsilon_i)$$

$$= \sum_{k=1}^{m+1} \varepsilon_k \prod_{i=1}^{k-1} (1-\varepsilon_i) \cdot \chi_k(j) + f^{(m+1)}_{\sigma^m(j)} \prod_{i=1}^{m+1} (1-\varepsilon_i)$$

$$\lambda_{N+m-1} = \sum_{k=1}^{m} \varepsilon_k \prod_{i=1}^{k-1} (1-\varepsilon_i) = \frac{1}{2^m}$$

$$c_k = \varepsilon_k \prod_{i=1}^{k-1} (1-\varepsilon_i) = \frac{1}{2^k} \quad \left(\varepsilon = \frac{1}{2}\right)$$

$$\Rightarrow \sum_{k=1}^{\infty} c_k = 1$$

$$\lambda_j = \sum_{k=1}^{\infty} \frac{1}{2^k} x_k(j)$$

$$x_k(j) = x_{[1,N]}(\sigma^{k-1}(j))$$

$j \in \{1, \dots, N-1\}$

$\sigma^{k-1}(j) = j \Rightarrow x_k(j) = 1, \quad \lambda_j = 1$

$j = N+k-1$

$\sigma^{k-1}(N+k-1) = N \Rightarrow x_k(j) = 1$

$j > N+k-1$

$$\sigma^{k-1}(j) = \underbrace{N + (j-N-k+1)}_{j-k+1} > N$$

$\Rightarrow x_k(j) = 0$

$x_k(j)$ undefined for $j \in \{N, N+1, \dots N+k-2\}$

but we should take $x_k(j) = 0$