Lemma

$$0 \le \varepsilon_j \le 1, \quad M \in \mathbb{N} \cup \{\infty\};$$
$$\sum_{k=1}^{M} \varepsilon_k \prod_{j=1}^{k-1} (1-\varepsilon_j) = 1 - \prod_{j=1}^{M} (1-\varepsilon_j)$$

Proof.

$$\sum_{k=1}^{M} (\alpha_{k-1} - \alpha_k) = \alpha_0 - \alpha_M$$ telescopic sum

$$\alpha_k = \begin{cases} \prod_{j=1}^{k} (1-\varepsilon_j) & (k=1,2,\dots,M) \\ 1 & (k=0) \end{cases}$$

$$\alpha_{k-1} - \alpha_k = \prod_{j=1}^{k-1} (1-\varepsilon_j) - (1-\varepsilon_k) \prod_{j=1}^{k-1} (1-\varepsilon_j)$$
$$= \varepsilon_k \prod_{j=1}^{k-1} (1-\varepsilon_j)$$

$$\alpha_0 - \alpha_1 = 1 - (1-\varepsilon_1) = \varepsilon_1$$

$$\varepsilon_k \prod_{j=1}^{k-1} (1-\varepsilon_j) \ge 0$$

$$\Rightarrow \sum_{k=1}^{M} \varepsilon_k \prod_{j=1}^{k-1} (1-\varepsilon_j) \le \sum_{k=1}^{M+1} \varepsilon_k \prod_{j=1}^{k-1} (1-\varepsilon_j) \le \cdots \le 1$$

$$\therefore \exists \lim_{M \to \infty} \sum_{k=1}^{M} \varepsilon_k \prod_{j=1}^{k-1} (1-\varepsilon_j) \qquad \forall M \in \mathbb{N} \cup \{\infty\} \quad //$$

Example

$$\lambda = \{\underbrace{1, \dots, 1}_{N-1}, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\} \in \widetilde{\mathcal{F}}_N$$

$$\varepsilon = \min\{\lambda_N, 1-\lambda_{N+1}\} = \min\{\frac{1}{2}, 1-\frac{1}{4}\} = \frac{1}{2}$$

$$T_\varepsilon : \lambda \mapsto f = \{\underbrace{1, \dots, 1}_{N-1}, 0, \frac{1}{2}, \frac{1}{2^2}, \dots\} \in \mathcal{F}_N$$

$$\begin{cases} f_N = \dfrac{\frac{1}{2}-\frac{1}{2}}{1-\frac{1}{2}} = 0 \\ f_{N+j} = \dfrac{\frac{1}{2^{j+1}}}{1-\frac{1}{2}} = \dfrac{1}{2^j} \end{cases}$$

$$\tau : f \mapsto \hat{f} = \{1, \dots, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\} \in \widetilde{\mathcal{F}}_N$$

$$\hat{f}_j = f_{\tau(j)}$$

$$\tau = \begin{pmatrix} 1 & \cdots & N-1 & N & N+1 & \cdots \\ 1 & \cdots & N-1 & N+1 & N+2 & \cdots \end{pmatrix}$$

$$\begin{array}{ccccc} \lambda & \xmapsto{T_\varepsilon} & f & \xmapsto{\tau} & \hat{f} \\ \in & & \in & & \in \\ \widetilde{\mathcal{F}}_N & & \mathcal{F}_N & & \widetilde{\mathcal{F}}_N \end{array}$$

Proof.

$$\gamma : L^2(\mathbb{R}^3;\mathbb{C}^q) \to L^2(\mathbb{R}^3;\mathbb{C}^q)$$

$$\gamma u_j = \lambda_j u_j \quad (j=1,2,\dots)$$

$$(u_j, u_k) = \delta_{jk}$$

$$\begin{cases} 0 \le \lambda_j \le 1, \quad \sum_{j=1}^{\infty} \lambda_j = N \\ \qquad\qquad\qquad\qquad \lambda = \{\lambda_j\}_{j=1}^{\infty} \in \mathcal{F}_N \\ \lambda_1 \ge \lambda_2 \ge \cdots \qquad\qquad \lambda \in \widetilde{\mathcal{F}}_N \end{cases}$$

$$\varepsilon = \min\{\lambda_N, 1-\lambda_{N+1}\} \Rightarrow \quad 0 < \varepsilon < 1$$

$$T_\varepsilon : \lambda \mapsto f$$

$$\begin{array}{ccc} \lambda & \mapsto & f \\ \cap & & \cap \\ \widetilde{\mathcal{F}}_N & \to & \mathcal{F}_N \end{array}$$

s.t. $\lambda_j = \varepsilon \chi_{[1,N]}(j) + (1-\varepsilon) f_j$

$$\Rightarrow \quad 0 \le f_j \le 1, \quad \sum_{j=1}^{\infty} f_j = N, \qquad f \in \mathcal{F}_N$$

$\tau : \mathbb{N} \to \mathbb{N}$ s.t. $\hat{f}_j = f_{\tau(j)}$

$$\hat{f}_1 \ge \hat{f}_2 \ge \cdots, \qquad \hat{f} \in \widetilde{\mathcal{F}}_N$$

$$\varepsilon_2 = \min\{\tilde{f}_N, 1-\tilde{f}_{N+1}\} \Rightarrow \quad 0 < \varepsilon_2 < 1$$

$$T_{\varepsilon_2} : \tilde{f} \longmapsto f^{(2)} \in \mathcal{F}$$

$$\text{s.t. } \tilde{f}_j = \varepsilon_2 \chi_{[1,N]}(j) + (1-\varepsilon_2) f^{(2)}_j$$

$$f_j = \tilde{f}_{\sigma(j)} = \varepsilon_2 \underbrace{\chi_{[1,N]}(\sigma(j))}_{= \chi_2(j)} + (1-\varepsilon_2) f^{(2)}_{\sigma(j)}$$

$$\sigma \simeq \tau^{-1}$$

$$\lambda_j = \varepsilon_1 \underbrace{\chi_{[1,N]}(j)}_{= \chi_1(j)} + f_j (1-\varepsilon_1)$$

$$= \varepsilon_1 \chi_1(j) + \left(\varepsilon_2 \chi_2(j) + f^{(2)}_{\sigma(j)} \right) (1-\varepsilon_1)$$

$$= \sum_{k=1}^{2} \varepsilon_k \prod_{i=1}^{k-1} (1-\varepsilon_i) \chi_k(j) + f^{(2)}_{\sigma(j)} \prod_{i=1}^{2} (1-\varepsilon_i)$$

$$\left\{ \begin{array}{ll} \varepsilon \to \varepsilon_1 & \\ \sigma \to \sigma_1 & \tau \to \tau_1 \\ f \to f^{(1)} & \text{etc...} \end{array} \right.$$

$$\tau_2 : \mathbb{N} \to \mathbb{N}, \quad \tilde{f}^{(2)}_j = f^{(2)}_{\tau_2(j)}$$

$$\tilde{f}^{(2)} \in \tilde{\mathcal{F}}_N$$

$$\varepsilon_3 = \min\left\{ \tilde{f}^{(2)}_{N},\ 1-\tilde{f}^{(2)}_{N+1} \right\} \Rightarrow 0<\varepsilon_3<1$$

$$T_{\varepsilon_3} : \tilde{f}^{(2)} \longmapsto f^{(3)} \in \mathcal{F}$$

$$\text{s.t. } \tilde{f}^{(2)}_{j} = \varepsilon_3 \chi_{[1,N]}(j) + (1-\varepsilon_3) f^{(3)}_{j}$$

$$f^{(2)}_{j} = \tilde{f}^{(2)}_{\sigma_2(j)}$$

$$\sigma_2 \simeq \tau_2^{-1}$$

$$f^{(2)}_{\sigma_1(j)} = \tilde{f}^{(2)}_{\sigma_2(\sigma_1(j))}$$

$$= \varepsilon_3 \underbrace{\chi_{[1,N]}(\sigma_2\sigma_1(j))}_{\chi_3(j)} + (1-\varepsilon_3) f^{(3)}_{\sigma_2\sigma_1(j)}$$

$$\therefore \lambda_j = \sum_{k=1}^{2} \varepsilon_k \prod_{i=1}^{k-1}(1-\varepsilon_i)\chi_k(j) + f^{(2)}_{\sigma_1(j)} \prod_{i=1}^{2}(1-\varepsilon_i)$$

$$= \sum_{k=1}^{3} \varepsilon_k \prod_{i=1}^{k-1}(1-\varepsilon_i)\chi_k(j) + f^{(3)}_{\sigma_2\sigma_1(j)} \prod_{i=1}^{3}(1-\varepsilon_i)$$

$$\tau_3 : \mathbb{N} \to \mathbb{N} \quad \text{s.t. } \tilde{f}^{(3)}_{j} = f^{(3)}_{\tau_3(j)}$$

$$\tilde{f}^{(3)}_{1} \geq \tilde{f}^{(3)}_{2} \geq \cdots, \quad \tilde{f}^{(3)} \in \tilde{\mathcal{F}}_N$$

$$\tilde{f}^{(M-1)} \in \tilde{\mathcal{F}}_N, \quad \tilde{f}_j^{(M-1)} = f^{(M-1)}_{\tau_{M-1}(j)}$$

$$\varepsilon_M = \min\{\tilde{f}_N^{(M-1)}, 1-\tilde{f}_{N+1}^{(M-1)}\} \Rightarrow 0 < \varepsilon_M < 1$$

$$T_{\varepsilon_M} : \tilde{f}^{(M-1)} \mapsto f^{(M)}$$

$$\text{s.t. } \tilde{f}_j^{(M-1)} = \varepsilon_M \chi_{[1,N]}(j) + (1-\varepsilon_M) f_j^{(M)}$$

$$f_j^{(M-1)} = \tilde{f}^{(M-1)}_{\sigma_{M-1}(j)}, \quad \sigma_{M-1} \simeq \tau_{M-1}^{-1}$$

$$\begin{aligned}
f^{(M-1)}_{\sigma_{M-2}\cdots\sigma_1(j)} &= \tilde{f}^{(M-1)}_{\sigma_{M-1}\sigma_{M-2}\cdots\sigma_1(j)} \\
&= \varepsilon_M \underbrace{\chi_{[1,N]}(\sigma_{M-1}\sigma_{M-2}\cdots\sigma_1(j))}_{\chi_M(j)} \\
&\qquad + (1-\varepsilon_M) f^{(M)}_{\sigma_{M-1}\cdots\sigma_1(j)}
\end{aligned}$$

$$\begin{aligned}
\therefore \lambda_j &= \sum_{k=1}^{M-1} \varepsilon_k \prod_{i=1}^{k-1}(1-\varepsilon_i)\chi_k(j) + f^{(M-1)}_{\sigma_{M-2}\cdots\sigma_1(j)} \prod_{i=1}^{M-1}(1-\varepsilon_i) \\
&= \sum_{k=1}^{M} \underbrace{\varepsilon_k \prod_{i=1}^{k-1}(1-\varepsilon_i)}_{C_k} \cdot \chi_k(j) + \underbrace{f^{(M)}_{\sigma_{M-1}\sigma_{M-2}\cdots\sigma_1(j)} \prod_{i=1}^{M}(1-\varepsilon_i)}_{R_j^{(M)}}
\end{aligned}$$

$$\tau_M : \mathbb{N} \to \mathbb{N} \text{ s.t. } \tilde{f}_j^{(M)} = f^{(M)}_{\tau_M(j)}$$

$$\tilde{f}^{(M)} \in \tilde{\mathcal{F}}_N.$$

$$\boxed{\sum_{k=1}^{\infty} c_k \leq 1}$$

$$\because) \quad \sum_{k=1}^{M} c_k = \sum_{k=1}^{M} \varepsilon_k \prod_{i=1}^{k-1} (1-\varepsilon_i)$$

$$= 1 - \prod_{i=1}^{M} (1-\varepsilon_i) \quad \leq 1$$

$$c_k \geq 0 \Rightarrow \sum_{k=1}^{M} c_k \text{ , non-decreasing}$$

$$\therefore \exists \lim_{M \to \infty} \sum_{k=1}^{M} c_k \leq 1 \quad //$$

$$d := \sum_{k=1}^{\infty} c_k$$

$$R_j^{(M)} = \underbrace{f^{(M)}_{\sigma_{M-1}\cdots\sigma_1(j)}}_{\overset{\shortparallel}{r_j^{(M)}}} \prod_{i=1}^{M} (1-\varepsilon_i) \geq 0$$

$$\boxed{\exists R = \{R_j\} \text{ s.t. } \lim_{M\to\infty} \sum_{j=1}^{\infty} \left| R_j^{(M)} - R_j \right| = 0}$$

$$\because)\ R_j^{(M)} = \lambda_j - \sum_{k=1}^{M} c_k x_k(j)$$

$$c_k x_k(j) \geq 0 \Rightarrow R_j^{(1)} \geq R_j^{(2)} \geq \cdots \geq R_j^{(M)} \geq \cdots \geq 0$$

$$\exists R_j \geq 0 \text{ s.t. } R_j^{(M)} \to R_j \geq 0 \quad (M\to\infty)$$

$$R_j = \lim_{M\to\infty} R_j^{(M)} = \lambda_j - \sum_{k=1}^{\infty} c_k x_k(j) = \lambda_j - d$$

$$R_j^{(M)} - R_j = d - \sum_{k=1}^{M} c_k x_k(j)$$

$$= \sum_{k=M+1}^{\infty} c_k x_k(j) \geq 0$$

$$\therefore \sum_{j=1}^{\infty} \left| R_j^{(M)} - R_j \right|$$

$$= \sum_{j=1}^{\infty} \sum_{k=M+1}^{\infty} c_k x_k(j)$$

$$= \sum_{k=M+1}^{\infty} c_k \underbrace{\sum_{j=1}^{\infty} x_k(j)}_{= N} \leq Nd < \infty$$

$$= N \Big(d - \underbrace{\sum_{k=1}^{M} c_k}_{\downarrow\, d} \Big) \longrightarrow 0 \quad (M \to \infty)$$

//

$$\boxed{\sum_{\bar{j}=1}^{\infty} R_{\bar{j}} = N(1-d)}$$

$\because)$ $R_{\bar{j}}^{(M)} = \lambda_{\bar{j}} - \sum_{k=1}^{M} c_k x_k(\bar{j})$

$$\sum_{\bar{j}=1}^{\infty} R_{\bar{j}}^{(M)} = \underbrace{\sum_{\bar{j}=1}^{\infty} \lambda_{\bar{j}}}_{\substack{\shortparallel \\ N}} - \underbrace{\sum_{\bar{j}=1}^{\infty} \sum_{k=1}^{M} c_k x_k(\bar{j})}_{\substack{\shortparallel \\ Nd}}$$

$$= N(1-d) \quad (\forall M = 1, 2, \ldots)$$

$$\therefore \sum_{\bar{j}=1}^{\infty} R_{\bar{j}} = \lim_{M \to \infty} \sum_{\bar{j}=1}^{\infty} R_{\bar{j}}^{(M)} \qquad \because R_{\bar{j}}^{(M)} \to R_{\bar{j}}$$

(strongly)

$$= \lim_{M \to \infty} N(1-d)$$

$$= N(1-d) \;_{//}$$

$$\boxed{R_j^{(m)} \leq 1 - \sum_{k=1}^{m} c_k}$$

$$:) \quad R_j^{(m)} + \sum_{k=1}^{m} c_k$$

$$= r_j^{(m)} \prod_{i=1}^{m} (1-\varepsilon_i) + \sum_{k=1}^{m} \varepsilon_k \prod_{i=1}^{k-1} (1-\varepsilon_i)$$

$$1 - \prod_{i=1}^{m} (1-\varepsilon_i)$$

$$= 1 - (1 - r_j^{(m)}) \prod_{i=1}^{m} (1-\varepsilon_i) \leq 1 \quad //$$

Cor. $\quad R_j = \lim_{m \to \infty} R_j^{(m)} \leq 1 - d$

$$\boxed{d < 1 \quad \Rightarrow \quad \exists \lim_{m \to \infty} \varepsilon_m > 0}$$

$$\because) \quad r_j^{(m)} = \frac{R_j^{(m)}}{1 - \sum_{k=1}^{m} c_k}$$

$$\begin{cases} R_j^{(m)} = \underbrace{f^{(m)}_{\sigma_{m-1} \cdots \sigma_1 (j)}}_{= \, r_j^{(m)}} \prod_{i=1}^{m} (1 - \varepsilon_i) \\ \sum_{k=1}^{m} c_k = 1 - \prod_{i=1}^{m} (1 - \varepsilon_i) \end{cases}$$

$$\Rightarrow \quad r_j^{(m)} = f^{(m)}_{\sigma_{m-1} \cdots \sigma_1 (j)} \qquad r^{(m)} \in \mathcal{F}_N$$

$$\exists \tau \text{ s.t. } \tilde{r}_j^{(m)} = r_{\tau(j)}, \quad \tilde{r}^{(m)} \in \tilde{\mathcal{F}}_N$$

$$\forall M \geq 1; \quad 1 - \tilde{r}_{N+1}^{(m)} \geq \frac{1}{N+1} > 0$$

$$\left(\exists \lim_{M \to \infty} \tilde{r}_{N+1}^{(m)} \; ? \right)$$

$$\begin{array}{ccccc} R_j^{(m)} & = & \left(1 - \sum_{k=1}^{m} c_k\right) & r_j^{(m)} & \\ \downarrow & & \downarrow & \downarrow & \\ R_j & & 1 - d & \frac{R_j}{1-d} & \xmapsto{\tau} \tilde{r}_j \end{array}$$

$$\tilde{r} \in \tilde{\mathcal{F}}_N$$

$$\tilde{r}_N = \frac{R_N}{1-d} > 0$$

$$\varepsilon_M = \min\left\{ \tilde{r}_N^{(M)},\ 1-\tilde{r}_{N+1}^{(M)} \right\}$$

$$\#\left\{ M \in \mathbb{N} \mid \varepsilon_M = 1-\tilde{r}_{N+1}^{(M)} \right\} \leq N < \infty$$

$$\lim_{M\to\infty} \varepsilon_M = \lim_{M\to\infty} \tilde{r}_N^{(M)}$$

$$= \lim_{M\to\infty} r_{\tau(N)}^{(M)}$$

$$= r_{\tau(N)} = \frac{R_{\tau(N)}}{1-d} > 0 \quad //$$

$$\sum_{k=1}^{M} \underbrace{\varepsilon_k \prod_{j=1}^{k-1}(1-\varepsilon_j)}_{= c_k} = 1 - \underbrace{\prod_{j=1}^{M}(1-\varepsilon_j)}_{\to 0}$$

$$c_k \to d$$

$d = 1$ contradiction!

$d=1$

$$\lambda_j = \sum_{k=1}^{\infty} c_k \chi_k(j)$$

$$\gamma = \sum_{j=1}^{\infty} \lambda_j \gamma_{u_j}$$

$$= \sum_j \sum_k c_k \chi_k(j) \cdot \gamma_{u_j}$$

$$= \sum_k c_k \underbrace{\sum_j \chi_k(j) \gamma_{u_j}}_{\text{rank } N} , \quad \sum_k c_k = 1$$

$$(\gamma\phi)(\underline{z}) = \sum_k c_k \sum_j \chi_k(j)\, (u_j, \phi)\, u_j(\underline{z})$$

$$\pi : \{1, \dots, N\} \to \mathbb{N}$$

$$\psi_\pi(\underline{z}) = \frac{1}{\sqrt{N!}} \det \{ u_{\pi(i)}(z_j) \}_{i,j=1}^{N}$$

$$\therefore \exists \Gamma = \Gamma_{\psi_\pi}, \quad (\Gamma_{\psi_\pi}\phi)(z) = (\psi_\pi, \phi)\psi_\pi$$

Finite iteration

$$\varepsilon_{M+1} = \min\{\hat{f}^{(M)}_N,\ 1-\hat{f}^{(M)}_{N+1}\} = 1$$

$$\Rightarrow \hat{f}^{(M)}_N = 1,\quad \hat{f}^{(M)}_{N+1} = 0$$

$$\hat{f}^{(M)}_j = \chi_{[1,N]}(j)$$

$$f^{(M)}_{\sigma_{M-1}\cdots\sigma_M(j)} = \hat{f}^{(M-1)}_{\sigma_M\sigma_{M-1}\cdots\sigma_1(j)}$$

$$= \underbrace{\chi_{[1,N]}(\sigma_M\sigma_{M-1}\cdots\sigma_1(j))}_{\chi_M(j)}$$

$$\therefore \lambda_j = \sum_{k=1}^{M}\varepsilon_k\prod_{i=1}^{k-1}(1-\varepsilon_i)\,\chi_k(j)$$

$$+ f^{(M)}_{\sigma_{M-1}\cdots\sigma_1(j)}\prod_{i=1}^{M}(1-\varepsilon_i)$$

$$= \sum_{k=1}^{M+1}\varepsilon_k\prod_{i=1}^{k-1}(1-\varepsilon_i)\cdot\chi_k(j)$$

$\longrightarrow$ Slater determinant