Theorem 3.2
(Admissible One-Body Density Matrices for Fermions)

γ on $\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^4)$ as in Theorem 3.1.

i.e. $(\phi', \gamma\phi) = (\gamma\phi', \phi)$ $\forall \phi, \phi' \in \mathcal{H}$ self-adjoint

$(\phi, \gamma\phi) \geq 0$ $\forall \phi \in \mathcal{H}$ positive semidefinite

$\mathrm{Tr}\,\gamma = N$, $N \geq 1$ integer

$\Longrightarrow$ $\exists \Gamma$: ~~bosonic~~ fermionic N-particle density matrix

s.t. $\gamma = N \cdot \mathrm{Tr}^{(N-1)} \Gamma$

$\Longleftrightarrow$ $\gamma \leq \mathbb{1}$.

Proof.

To ~~show~~ prove

$\gamma \leq \mathbb{1} \implies \exists \Gamma$

$1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ in decreasing order

$\gamma u_j = \lambda_j u_j$ for $j = 1, 2, \cdots$

$(u_j, u_k) = \delta_{jk}$

$\lambda_j = \lambda_j (u_j, u_j) = (u_j, \lambda_j u_j) = (u_j, \gamma u_j) \geq 0$

$\lambda_1 = (u_1, \gamma u_1) \leq 1$

$\sum_{j=1}^{\infty} \lambda_j = \mathrm{Tr}\, \gamma = N.$

Case I. $\lambda_{N+1} = 0$

$\implies \lambda_1 = \lambda_2 = \cdots = \lambda_N = 1, \quad \lambda_j = 0$ for $j \geq N+1$

Slater determinant

$\psi(\underline{z}) = (N!)^{-1/2} \det \{ u_i(z_j) \}_{i,j=1}^{N}$

with $u_1, \ldots, u_N$

$\Gamma = \Gamma_\psi$ with $\gamma = N \cdot \mathrm{Tr}_{(N-1)} \Gamma$

$$\Gamma_\psi(\underline{z};\underline{z}') = \psi(z)\overline{\psi(z')}$$

$$\Gamma_\psi \phi = (\psi, \phi)\psi = \int \overline{\psi(z')}\,\phi(z')\,\psi(z)\,dz'$$

$$= \int \underbrace{\psi(z)\overline{\psi(z')}}_{\Gamma(z;z')} \cdot \phi(z')\,dz'$$

$$\gamma^{(1)}(z_1;z_1') = N\cdot\int \psi(z_1,\underline{z}^{(N-1)})\,\overline{\psi(z_1',\underline{z}^{(N-1)})}\,d\underline{z}^{(N-1)}$$

$$\psi(z_1,\underline{z}^{(N-1)}) = (N!)^{-1/2}\sum_{\pi\in S_N}\varepsilon_\pi \prod_{i=1}^{N} u_i(z_{\pi(i)})$$

$$\prod_{i=1}^{N} u_{\pi(i)}(z_i)$$

$$= (N!)^{1/2}\sum_{\pi\in S_N}\varepsilon_\pi\, u_{\pi(1)}(z_1)\prod_{i=2}^{N} u_{\pi(i)}(z_i)$$

$$\gamma^{(1)}(z_1;z_1')$$

$$= \frac{1}{(N-1)!}\sum_{\pi\in S_N}\sum_{\pi'\in S_N}\varepsilon_\pi\varepsilon_{\pi'}\int u_{\pi(1)}(z_1)\,\overline{u_{\pi'(1)}(z_1')}$$

$$\times\int\prod_{i=2}^{N} u_{\pi(i)}(z_i)\;\overline{u_{\pi'(i)}(z_i)}\;d\underline{z}^{(N-1)}$$

$$\prod_{i=2}^{N}\int u_{\pi(i)}(z_i)\,u_{\pi'(i)}(z_i)\,dz_i$$

$$\prod_{i=2}^{N}(u_{\pi(i)}, u_{\pi'(i)}) = 1$$

if $\pi(i)=\pi'(i)$, for $i=2,\dots,N$

$\Rightarrow \pi(1)=\pi'(1)$

$\therefore \pi=\pi'$

$\therefore \gamma^{(1)}(z\,;z')$

$$= \frac{1}{(N-1)!} \sum_{\substack{\pi \in S_N \\ \pi = \pi'}} \varepsilon_\pi^2 \, u_{\pi(1)}(z) \overline{u_{\pi(1)}(z')}$$

$$\sum_{\pi \in S_N} = \sum_{j=1}^{N} \sum_{\pi(1)=j}$$

$$= \frac{1}{(N-1)!} \sum_{j=1}^{N} \sum_{\pi(1)=j} u_j(z) \overline{u_j(z')}$$

$$= \frac{1}{(N-1)!} \underbrace{\#\{\pi \in S_N \,;\, \pi(1)=j\}}_{(N-1)!} \left(\sum_{j=1}^{N}\right) u_j(z) \overline{u_j(z')}$$

$$= \sum_{j=1}^{N} u_j(z) \overline{u_j(z')} \,. \quad //$$

Case II. $\lambda_{N+1} > 0$

$\cancel{\lambda_{N+1} \leq \frac{N}{N+1} < 1}$ $\quad \boxed{0 < \lambda_{N+1} < 1}$

$\because)\ N = \sum_{j=1}^{\infty} \lambda_j \geq \sum_{j=1}^{N+1} \lambda_j \geq (N+1)\lambda_{N+1}$ //

$\therefore\ \lambda_{N+1} \leq \frac{N}{N+1}$

$\boxed{0 < \lambda_N < 1.}$

$1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$

$\lambda_N = 0 \implies \lambda_j = 0$ for $j \geq N+1$

$\sum_{j=1}^{\infty} \lambda_j = \sum_{j=1}^{N-1} \lambda_j \leq N-1 \neq N$

Contradiction!

$\lambda_N = 1 \implies 1 \geq \lambda_1 \geq \cdots \geq \lambda_N = 1$

$\therefore\ \lambda_1 = \cdots = \lambda_N = 1.$

$\implies \lambda_{N+1} = 0$ contradiction!

$$\chi_{[1,N]}(j) = \begin{cases} 1 & (1 \leq j \leq N) \\ 0 & (j > N) \end{cases}$$

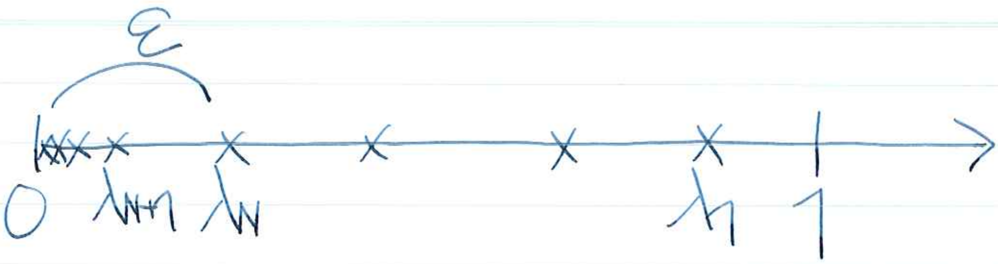
$\varepsilon = \min\{\lambda_N, 1 - \lambda_{N+1}\}$

$0 < \varepsilon < 1.$

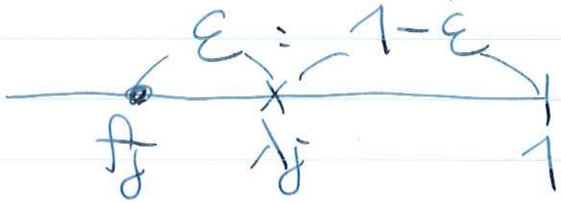
$\lambda_j = \varepsilon \chi_{[1,N]}(j) + (1-\varepsilon) f_j$

with $f_j = \frac{1}{1-\varepsilon}(\lambda_j - \varepsilon \chi_{[1,N]}(j))$

$\implies 0 \leq f_j \leq 1$ and $\sum_{j=1}^{\infty} f_j = N.$



a) $1 \le j \le N$



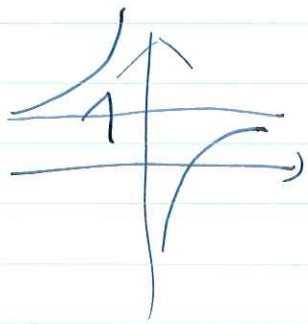
$$0 \le f_j = \frac{\lambda_j - \varepsilon}{1-\varepsilon} \overset{\le}{\cancel{\ge}} \lambda_j$$

$$\lambda_j = \varepsilon + (1-\varepsilon) f_j \le \varepsilon f_j + (1-\varepsilon) f_j = f_j$$

$$\frac{b+x}{a+x} = \frac{(a+x)-(a-b\cancel{+x})}{a+x} = 1 \cancel{+} - \frac{a-b+x}{a+x}$$

$$= 1 - \frac{a-b}{a+x}$$

$$\frac{a-b}{(a+x)^2} \ge 0$$



$$a \ge b$$

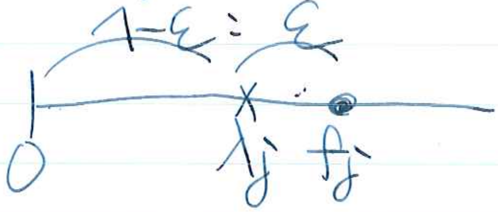
$$f_N = \frac{\lambda_N - \varepsilon}{1-\varepsilon} = \frac{\lambda_N - \lambda_N}{1-\lambda_N} = 0$$

$\cancel{f_1}$

$$0 \le f_j < \lambda_j$$

$$0 = f_N \le f_{N-1} \le \cdots \le f_1 .$$

a $j \geq N+1$

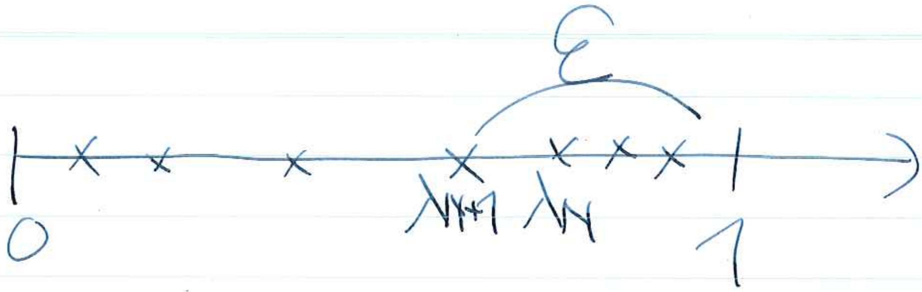


$$\lambda_j = (1-\varepsilon) f_j, \quad f_j = \frac{\lambda_j}{1-\varepsilon} \geq \frac{\lambda_j}{1-\varepsilon+\varepsilon} = \lambda_j$$

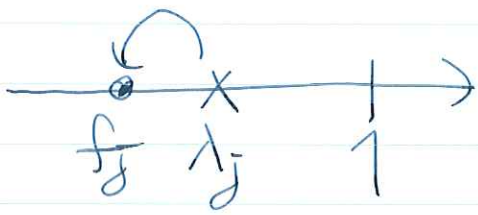
$$\lambda_j = 0 \implies f_j = 0$$

$$\begin{array}{ccccccc} f_{N+1} & \geq & f_{N+2} & \geq & \cdots & \geq & 0 \\ \vee\!| & & \vee\!| & & & & \\ \lambda_{N+1} & & \lambda_{N+2} & & & & \end{array}$$

$$f_j = \frac{\lambda_j}{1-\varepsilon} \leq \frac{\lambda_j}{1-(1-\lambda_N)} = \frac{\lambda_j}{\lambda_N} \leq 1.$$



@ $1 \le j \le N$



$f_j = \dfrac{\lambda_j - \varepsilon}{1-\varepsilon}$

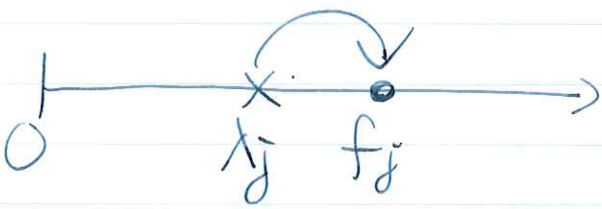
$\cancel{\lambda_j \le} f_j \le 1.$ $\quad \because f_j = \dfrac{\lambda_j - \varepsilon}{1-\varepsilon} \le \dfrac{1-\varepsilon}{1-\varepsilon} = 1$

$1 \ge f_1 \ge f_2 \ge \cdots \ge f_N$

$\lambda_j = \varepsilon + (1-\varepsilon) f_j$
$> \varepsilon f_j + (1-\varepsilon) f_j$
$= f_j$

$0 \le f_j < \lambda_j \le 1$

@ $j \ge N+1$



$f_j = \dfrac{\lambda_j}{1-\varepsilon}$

$0 \le \lambda_j \le f_j \le 1$

$f_j = \dfrac{\lambda_j}{1-\varepsilon} = \dfrac{\lambda_j}{1-(1-\lambda_N)} = \dfrac{\lambda_j}{\lambda_N} \le 1$

$\cancel{ff}\ \lambda_j = (1-\varepsilon) f_j \le f_j$

$f_{N+1} = \dfrac{\lambda_{N+1}}{1-(1-\lambda_{N+1})} = 1.$

$1 = f_{N+1} \ge f_{N+2} \ge \cdots \ge 0$