# Trace of Operators

$$A : \mathcal{H} \to \mathcal{H}$$

<u>Def.</u> $\operatorname{Tr} A = \sum_j (\phi_j, A\phi_j)$

$\forall \{\phi_j\}$ : ONB in $\mathcal{H}$

$\Leftrightarrow (\phi_j, \phi_k) = \delta_{jk}$

(iii) $A \geq 0 \Rightarrow \sum_j (\phi_j, A\phi_j) = \sum_j (\phi_j', A\phi_j')$

where $\{\phi_j\}, \{\phi_j'\}$ : ONB in $\mathcal{H}$

$\because$) $A \geq 0 \Rightarrow \exists \sqrt{A} : \mathcal{H} \to \mathcal{H}$

s.t. $A = (\sqrt{A})^* \sqrt{A}$

$$\begin{aligned}
&\sum_j (\phi_j', A\phi_j') \\
&= \sum_j \| \sqrt{A}\phi_j' \|^2 \\
&= \sum_j \sum_k |(\sqrt{A}\phi_j', \phi_k)|^2 \\
&= \sum_k \sum_j |(\phi_j', \sqrt{A}\phi_k)|^2 \\
&= \sum_k \| \sqrt{A}\phi_k \|^2 = \sum_k (\phi_k, A\phi_k) \quad //
\end{aligned}$$

Parseval

$\|f\|^2 = \sum_k |(f, \phi_k)|^2$

eigenvalues $\{\lambda_j\}$ corresponding eigenfunctions $\{\phi_j\}$

$$A\phi_j = \lambda_j \phi_j$$

$A \geq 0$, self-adjoint $\Longrightarrow$ $\lambda_j \geq 0 \quad \forall j \in \mathbb{N}$

$\because)$ $(A\phi_j, \phi_j) = (\lambda_j \phi_j, \phi_j)$

$$= \overline{\lambda_j}(\phi_j, \phi_j)$$

$$= \overline{\lambda_j}$$

Similarly, $(\phi_j, A\phi_j) = \lambda_j$

$A$ : self-adjoint $\Longleftrightarrow$ $(A\phi_j, \phi_j) = (\phi_j, A\phi_j)$

$$\therefore \overline{\lambda_j} = \lambda_j$$

$$\therefore \lambda_j \in \mathbb{R}$$

$\lambda_j = \lambda_j(\phi_j, \phi_j) = (\phi_j, \lambda_j\phi_j)$

$= (\phi_j, A\phi_j) \geq 0$

positive semidefinite //

$$\Gamma_\psi \psi = (\psi, \psi)\psi = 1 \cdot \psi$$

$$\Gamma_\psi \phi_j = \lambda_j \phi_j$$

$$\{\phi_j\} = \{\psi\} \cup \{\phi_j\}_{j \geq 1}$$

$$(\phi_j, \psi) = 0 \quad \forall j \geq 1$$

$$(\phi_j, \psi) = 0 \Rightarrow \Gamma_\psi \phi_j = 0$$

$$\begin{aligned} \therefore \operatorname{Tr}\Gamma_\psi &= \sum_j (\phi_j, \Gamma_\psi \phi_j) \\ &= \underbrace{(\psi, \psi)}_{=1} + \sum_{j \geq 1} \underbrace{(\phi_j, \Gamma_\psi \phi_j)}_{=0} \\ &= 1 \,/\!/ \end{aligned}$$

On the other hand,

$$\begin{aligned} 1 = \operatorname{Tr}\Gamma_\psi &= \sum_j (\phi_j, \Gamma_\psi \phi_j) \\ &= \sum_j (\phi_j, \lambda_j \phi_j) \\ &= \sum_j \lambda_j (\phi_j, \phi_j) = \sum_j \lambda_j \end{aligned}$$

$$\lambda_0 = 1, \quad \lambda_j \geq 0 \quad (\forall j \geq 1)$$

$$\Rightarrow \lambda_j = 0 \quad (\forall j \geq 1)$$

all other eigenvalues are zero!