Slater determinant

$$\psi(\underline{z}) = (N!)^{-1/2} \det\{u_i(z_j)\}_{i,j=1}^{N}$$

$$= \frac{1}{\sqrt{N!}} \begin{vmatrix} u_1(z_1) & \cdots & u_1(z_N) \\ \vdots & \ddots & \vdots \\ u_N(z_1) & \cdots & u_N(z_N) \end{vmatrix}$$

$$= \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} \varepsilon_\pi \prod_{i=1}^{N} u_i(z_{\pi(i)})$$

where $u_i \in L^2(\mathbb{R}^3; \mathbb{C}^2)$ for $i = 1, \cdots, N$

with $(u_j, u_k) = \int \overline{u_j(z)} u_k(z)\, dz$

$$= \delta_{jk}$$

Then, ψ is antisymmetric and normalized.

Rem.

$$\varepsilon_\pi = \operatorname{sgn}\pi = \begin{cases} +1 & \pi \text{ is even} \\ -1 & \pi \text{ is odd} \end{cases}$$

for a permutation $\pi \in S_N$

Kronecker delta

$$\delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

(1) antisymmetric

$$\psi(z_1, \dots, \overset{j}{\check{z}}_K, \dots, \overset{k}{\check{z}}_J, \dots, z_N)$$

$$= \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} \mathrm{sgn}\begin{pmatrix} 1 & \cdots & j & \cdots & k & \cdots & N \\ \pi(1) & \cdots & K & \cdots & J & \cdots & \pi(N) \end{pmatrix} \cdots u_j(z_K) \cdots u_k(z_J) \cdots$$

$$\pi(j) = K, \quad \pi(k) = J$$

$$= \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} \mathrm{sgn}\left[(J\,K)\begin{pmatrix} \cdots & j & \cdots & k & \cdots \\ \cdots & J & \cdots & K & \cdots \end{pmatrix}\right] \cdots u_j(z_J) \cdots u_k(z_K) \cdots$$

$$\mathrm{sgn}(J\,K) = -1$$

$$= -\frac{1}{\sqrt{N!}} \sum_{\pi' \in S_N} \mathrm{sgn}\begin{pmatrix} \cdots & j & \cdots & k & \cdots \\ \cdots & J & \cdots & K & \cdots \end{pmatrix} \cdots u_j(z_{\pi'(j)}) \cdots u_k(z_{\pi'(k)}) \cdots$$

$$\pi' = (J\,K)\,\pi$$

$$\pi'(j) = J, \quad \pi'(k) = K$$

$$= -\psi(z_1, \dots, z_J, \dots, z_K, \dots, z_N) \quad //$$

Q.E.D.

(2) normalized

$$\|\psi\|_2^2 = (\psi, \psi) = \int \overline{\psi(\underline{z})}\,\psi(\underline{z})\, d\underline{z}$$

$$\psi(\underline{z}) = \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} \varepsilon_\pi \prod_{j=1}^{N} u_j(z_{\pi(j)})$$

$$= \frac{1}{N!} \sum_{\pi \in S_N} \sum_{\pi' \in S_N} \varepsilon_{\pi\pi'} \underbrace{\int \prod_{j=1}^{N} \overline{u_j(z_{\pi(j)})} \prod_{k=1}^{N} u_k(z_{\pi'(k)})\, d\underline{z}}_{= I}$$

$$I = \int \prod_{J=1}^{N} \overline{u_{\pi^{-1}(J)}(z_J)} \prod_{K=1}^{N} u_{\pi'^{-1}(K)}(z_K)\, d\underline{z}$$

$$= \int \prod_{J=1}^{N} \overline{u_{\pi^{-1}(J)}(z_J)}\, u_{\pi'^{-1}(J)}(z_J)\, d\underline{z}$$

$$= \prod_{J=1}^{N} \int \overline{u_{\pi^{-1}(J)}(z_J)}\, u_{\pi'^{-1}(J)}(z_J)\, dz_J$$

$$= \prod_{J=1}^{N} \left(u_{\pi^{-1}(J)}, u_{\pi'^{-1}(J)} \right)$$

Here, $\pi^{-1}(J) = \pi'^{-1}(J)$ for $\forall J = 1, \dots, N$

$$\Leftrightarrow \pi^{-1} = \pi'^{-1} \Leftrightarrow \pi = \pi'$$

$$\sum_{\pi} \sum_{\pi'} = \sum_{\pi} \Big(\sum_{\pi'=\pi} + \underbrace{\sum_{\pi' \neq \pi}}_{=0} \Big)$$

$$\therefore \|\psi\|_2^2 = \frac{1}{N!} \sum_{\substack{\pi \in S_N \\ (\pi'=\pi)}} (\operatorname{sgn}\pi)^2 \prod_{j=1}^{N} (u_j, u_j)$$

$$= \frac{1}{N!} \sum_{\pi} 1$$

$$= 1 \quad (\because |S_N| = N!) \,//$$

Q.E.D.