Inequalities in Stability of Matter

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Spectral analysis of Schrödinger operators

- Schrödinger operators

$$H = -\Delta + V(x) \quad \text{in } L^2(\mathbb{R}^d)$$

where Δ denotes Laplacian in $\mathbb{R}^d$ and V is a real-valued function of $\mathbb{R}^d$.

- Non-positive eigenvalues

$$\{E_j\} = \text{all non-positive eigenvalues of } H$$

Lieb-Thirring inequalities (1976)

Let $\gamma \geq 0$. Assume that the negative part of the potential $V_{-}(x) = \max\{-V(x), 0\}$ satisfies the condition $V_{-} \in L^{\gamma+d/2}(\mathbb{R}^d)$. Then, there is a constant $L_{\gamma,d} > 0$ for suitable d defined below which is independent of V such that

$$\sum_j |E_j|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_{-}(x)^{\gamma+d/2}\, dx. \tag{1}$$

This holds in the following cases:

$$\begin{aligned} &\gamma \geq 1/2 && \text{for } d = 1, \\ &\gamma > 0 && \text{for } d = 2, \\ &\gamma \geq 0 && \text{for } d \geq 3. \end{aligned} \tag{2}$$

Otherwise, there is V that violates (1) for any finite choice of $L_{\gamma,d}$.

History - Recent development

We are interesting in

$$R_{\gamma,d} = L_{\gamma,d} / L^{\mathrm{cl}}_{\gamma,d} \tag{1}$$

where

$$L^{\mathrm{cl}}_{\gamma,d} = (2\pi)^d \int_{|p|\leq 1} (1-|p|^2)^\gamma \, dp = \frac{2(2\pi)^{-d}|S^{d-1}|}{d(d+2)} \tag{2}$$

only for $\gamma = 1$

with $|S^{d-1}|$ meaning the surface area of the unit ball in $\mathbb{R}^d$.

- Dolbeault, Laptev and Loss (2008)

$$R_{1,d} \leq \pi/\sqrt{3} = 1.81...$$

- Rumin (2011), Solovej (2011)

$$R_{1,d} \leq \left(\frac{d+4}{d}\right)^{d/2}$$

Cut-off wavefunction in momentum space

For $\phi \in L^2(\mathbb{R}^d)$

$$\phi^\varepsilon(x) = \mathscr{F}^{-1}\left[\chi_{[0,\varepsilon)}(|p|^2)(\mathscr{F}\phi)(p)\right], \tag{1}$$

where $\mathscr{F}$ stands for Fourier transform and χ denotes the characteristic function defined by

$$\chi_{[0,\varepsilon)}(|p|^2) = \begin{cases} 1 & \text{if } 0 \le |p|^2 < \varepsilon \\ 0 & \text{if } \varepsilon \le |p|^2 \end{cases} . \tag{2}$$

Lemma 1

For every $\phi \in H^1$

$$\int_{\mathbb{R}^d} |\nabla \phi(x)|^2 \, dx = \int_{\mathbb{R}^d} \int_0^\infty |\phi(x) - \phi^\varepsilon(x)|^2 \, d\varepsilon \, dx.$$

Lemma 2

For any sequence $\{\phi_j\}_j \subset L^2(\mathbb{R}^d)$

$$\Big(\sum_j |\phi_j(x) - \phi_j^{\varepsilon}(x)|^2\Big)^{1/2} \geq \left[\Big(\sum_j |\phi_j(x)|^2\Big)^{1/2} - \Big(\sum_j |\phi_j^{\varepsilon}(x)|^2\Big)^{1/2}\right]_+ ,$$

where $[f]_+$ denotes the positive part of f.

Lemma 3

Let $\{\phi_j\}_j$ be an orthonormal system in $L^2(\mathbb{R}^d)$. Then

$$\sum_j |\phi_j^\varepsilon(x)|^2 \leq (2\pi)^{-d} d^{-1} |S^{d-1}| \varepsilon^{d/2},$$

where $|S^{d-1}|$ denotes the surface area of the unit ball in $\mathbb{R}^d$.

Proposition for kinetic energy [Solovej]

Let $\{\phi_j\}_j$ be an orthonormal system in $L^2(\mathbb{R}^d)$. Then

$$\sum_j \int_{\mathbb{R}^d} |\nabla \phi_j(x)|^2 \, dx \geq \frac{(2\pi)^2 d^{2+2/d} |S^{d-1}|^{-2/d}}{(d+2)(d+4)} \int_{\mathbb{R}^d} \Big(\sum_j |\phi_j(x)|^2\Big)^{1+2/d} \, dx.$$

Solovej's approach

$$\sum_j \int_{\mathbb{R}^d} |\nabla \phi_j(x)|^2 \, dx \geq \int_{\mathbb{R}^d} \int_0^\infty \left[A_0 - B\varepsilon^{d/4}\right]_+^2 \, d\varepsilon dx \tag{1}$$

where

$$A_0 = \left(\sum_j |\phi_j(x)|^2\right)^{1/2} \tag{2}$$

and

$$B = (2\pi)^{-d/2} d^{-1/2} |S^{d-1}|^{1/2}. \tag{3}$$

Proposed idea

$$\int_0^\infty \left[(\sum_j |\phi_j(x)|^2)^{1/2} - (\sum_j |\phi_j^\varepsilon(x)|^2)^{1/2} \right]_+^2 d\varepsilon \geq \sum_{n=0}^{\infty} \int_{\varepsilon_n}^{\varepsilon_{n+1}} \{A_n - B(\varepsilon - \varepsilon_n)^{d/4}\}^2 \, d\varepsilon,$$

where

$$A_n = A_0 - \left(\sum_j |\phi_j^{\varepsilon_n}(x)|^2 \right)^{1/2}$$

and $\varepsilon_{n+1} - \varepsilon_n = (A_n/B)^{4/d}$ **with** $\varepsilon_0 = 0$. **Then**

$$\int_{\varepsilon_n}^{\varepsilon_{n+1}} \{A_n - B(\varepsilon - \varepsilon_n)^{d/4}\}^2 \, d\varepsilon = \frac{d^2}{(d+2)(d+4)} B^{-4/d} A_n^{2+4/d}.$$

Conjecture

There is $\kappa > 0$ independent of x and n such that

$$(1-\kappa^n)^2 \sum_j |\phi_j^{\varepsilon_n}(x)|^2 \le \sum_j |\phi_j(x)|^2. \tag{1}$$

for any $x \in \mathbb{R}^d$ and $n = 0, 1, 2, \ldots.$

Expected results

$$\begin{aligned}\sum_j E_j &= \sum_j (\phi_j, H\phi_j) \\ &\geq -\frac{2(2\pi)^{-d}|S^{d-1}|}{d(d+2)}\left(\frac{(d+4)(1-\kappa)}{d}\right)^{d/2}\int_{\mathbb{R}^d} V_-(x)^{1+d/2}\,dx\end{aligned} \tag{1}$$

with $0 \leq \kappa \leq 4/(d+4)$. We have just obtained the new estimate in the Lieb-Thirring inequalities for $\gamma = 1$, and

$$R_{1,d} \leq \left(\frac{(d+4)(1-\kappa)}{d}\right)^{d/2}. \tag{2}$$

Application

- Lieb-Thirring inequalities can be applied
  - to estimate the ground state energy of quantum systems
    - Lieb and Thrring (1976)
  - to estimate dimensions of attractors in theory of Navier-Stokes equations
    - Lieb (1984)
  - to prove geometrical problem for ovals in the plane
    - Benguria and Loss (2004)

Concluding Remarks

- Lieb-Thirring inequalities play an essential role in stability of quantum systems
- Solovej's previous approach is effective in proof of the inequalities
- If the proposed conjecture is true, we have an improvement of the inequalities