Recent progress in Lieb-Thirring inequalities

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Settings

The operator for quantum systems

Definition (Schrödinger operators)

Let

$$H = -\triangle + V(x) \quad \text{on } L^2(\mathbb{R}^d),$$

where $\triangle$ denotes the Laplace operator in $\mathbb{R}^d$
and V is a real-valued potential function of $x \in \mathbb{R}^d$.

Our Goal

Energy eigenvalues

$E_0, E_1, E_2, \ldots$ denote all non-positive eigenvalues of H.

The Riesz mean

$$\sum_j |E_j|^\gamma$$

should be estimated in the view of V. In particular,

$$\sum_j |E_j|^\gamma = \begin{cases} \text{the number of eigenvalues} & \text{for } \gamma = 0 \\ \text{the sum of possible energies} & \text{for } \gamma = 1. \end{cases}$$

Lieb-Thirring inequalities

which is the well known result in mathematical physics

Theorem (Lieb and Thirring, 1976)

Let $\gamma \geq 0$. Assume that $V_-(x) = \max\{-V(x), 0\}$ satisfies the condition $V_- \in L^{\gamma+d/2}(\mathbb{R}^d)$. Then, there is $L_{\gamma,d} > 0$ which is independent of V such that

$$\sum_j |E_j|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-(x)^{\gamma+d/2}\, dx \tag{1}$$

holds when $\gamma \geq 1/2$ for $d = 1$, $\gamma > 0$ for $d = 2$, and $\gamma \geq 0$ for $d \geq 3$. Otherwise, there is V that violates the inequalities (1) for any finite choice of $L_{\gamma,d}$.

Remarks on Lieb-Thirring inequalities

- Lieb and Thirring [9]; almost all cases in Lieb-Thirring inequalities. (1976)
- Cwikel [1], Lieb [7] and Rozenbljum [10]; the critical case $\gamma = 0$ for $d \geq 3$. (1970s)
- Weidl [13]; the remaining case $\gamma = 1/2$ for $d = 1$. (1996)

Semi-classical approximation

The coefficients $L_{\gamma,d}$ should be compared to classical ones obtained by the semi-classical approximation

$$L^{\mathrm{cl}}_{\gamma,d} = (2\pi)^d \int_{|p|\leq 1} (1-|p|^2)^\gamma \, dp = \frac{\Gamma(\gamma+1)}{(4\pi)^{d/2}\Gamma(\gamma+1+d/2)}$$

where Γ is the Gamma function.
It is also known that $L_{\gamma,d}/L^{\mathrm{cl}}_{\gamma,d} \geq 1$ for all possible γ and d, and that $L_{\gamma,d}/L^{\mathrm{cl}}_{\gamma,d}$ is non-increasing on γ. (Aizenman and Lieb, 1978)

Semi-classical approximation

- Helffer and Robert [3]; $L_{\gamma,d}/L^{\mathrm{cl}}_{\gamma,d} > 1$ for $\gamma < 1$. (1990)
- Hundertmark, Lieb and Thomas [5]; $L_{1/2,1} = 2L^{\mathrm{cl}}_{1/2,1}$. (1998)
- Hundertmark, Laptev and Weidl [4]; $L_{\gamma,1} = L^{\mathrm{cl}}_{\gamma,1}$ for $\gamma \geq 3/2$. (2000)
- Laptev and Weidl [6] enables that $L_{\gamma,d} = L^{\mathrm{cl}}_{\gamma,d}$ if $\gamma \geq 3/2$ for all d. (2000)

Recent results (1)

Dolbeault, Laptev and Loss [2] have improved the coefficient which is also known as best possible at the present time.

Dolbeault, Laptev and Loss, 2008

$$L_{1,d}/L^{\mathrm{cl}}_{1,d} \le \pi/\sqrt{3} = 1.81\ldots \quad \text{for all } d.$$

[2] J. Dolbeault, A. Laptev and M. Loss; J. Eur. Math. Soc., Vol. 10, pp. 1121-1126. (2008)

Recent results (2)

Rumin [11] and Solovej [12] has proposed a new approach of proving that

Rumin and Solovej, 2011

$$L_{1,d}/L_{1,d}^{\mathrm{cl}} \leq \left(\frac{d+4}{d}\right)^{d/2}.$$

[11] M. Rumin, Duke Math. J., **160**, no. 3, 567–597. (2011)
[12] J. P. Solovej,
"The Lieb-Thirring inequality." (2011)

Lieb-Thirring conjecture

It is conjectured by Lieb and Thirring [9] that the optimal $L_{1,3}$ coincides with $L^{\mathrm{cl}}_{1,3}$.

Conjecture (Lieb and Thirring, 1976)

$$L_{1,3} = L^{cl}_{1,3}$$

[9] E. H. Lieb and W. E. Thirring, "Inequalities for the Moments of the Eigenvalues of the Schrödinger Hamiltonian and their Relation to Sobolev Inequarities," in *Studies in Mathematical Physics* (1976), pp. 269–303.

Stability of Matter

$$H = \frac{1}{2}\sum_{j=1}^{N}\left(-i\nabla_j + \sqrt{\alpha}A(x_j)\right)^2 + \alpha V(X, R),$$

where $\alpha > 0$ is Sommerfeld's fine structure constant, A is an arbitrary magnetic vector potential in $L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$. The Coulomb potential is written by

$$V(X, R) = \sum_{j=1}^{N}\sum_{k=j+1}^{N}\frac{1}{|x_j - x_k|} - \sum_{j=1}^{N}\sum_{k=1}^{M}\frac{Z_k}{|x_j - R_k|} + \sum_{j=1}^{M}\sum_{k=j+1}^{M}\frac{Z_j Z_k}{|R_j - R_k|},$$

where M is the number of nucleon.

Stability of Matter

Why is our world stable?

Lieb and Thirring [9] have improved the result by Dyson and Lenard for the stability of non-relativistic matter.

Theorem (Stability of matter of the second kind)

Let $Z_{max} = \max_j\{Z_j\}$. For all normalized, antisymmetric wavefunction ψ with q spin states,

$$(\psi, H\psi) \geq -0.747\alpha^2 N q^{2/3} \left(1 + 2.56 Z_{max}(M/N)^{1/3}\right)^2 .$$

Energy cutoff method

For $\phi \in L^2(\mathbb{R}^d)$ we use

$$\phi^\varepsilon(x) = \mathcal{F}^{-1}\left[\chi_{[0,\varepsilon)}(|p|^2)\hat{\phi}(p)\right],$$

where $\mathcal{F}^{-1}$ is the Fourier inverse transform and χ denotes the characteristic function

$$\chi_{[0,\varepsilon)}(x) = \begin{cases} 1 & \text{if } 0 \le x < \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

New results

Condition (sufficient)

Let ϕ_j be the eigenfunction corresponding to the eigenvalue E_j of H. Then, there is $\kappa > 0$ independent of x and n such that

$$(1-\kappa^n)^2 \sum_j |\phi_j^{\varepsilon_n}(x)|^2 \leq \sum_j |\phi_j(x)|^2 \quad a.\ e.,$$

where $\varepsilon_n \geq 0$ are some increasing sequence.

Main result

We obtain the new estimate of the coefficients for $\gamma = 1$ if the condition is true.

Theorem

If the condition above is true, we can improve the estimate of the coefficients

$$L_{1,d}/L_{1,d}^{cl} \leq \left(\frac{(d+4)(1-\tilde{\kappa})}{d} \right)^{d/2}$$

with $0 \leq \tilde{\kappa} \leq 4/(d+4)$ *and* $\tilde{\kappa}$ *is a monotonic decreasing function of* κ.

Lemma 1

Lemma

For every $\phi \in H^1$

$$\int_{\mathbb{R}^d} |\nabla \phi(x)|^2 \, dx = \int_{\mathbb{R}^d} \int_0^\infty |\phi(x) - \phi^\varepsilon(x)|^2 \, d\varepsilon dx.$$

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla \phi(x)|^2 \, dx &= \int_{\mathbb{R}^d} |p|^2 |\hat{\phi}(p)|^2 dp \\ &= \int_{\mathbb{R}^d} \int_0^{|p|^2} |\hat{\phi}(p)|^2 \, d\varepsilon dp \\ &= \int_{\mathbb{R}^d} \int_0^\infty (1 - \chi_{[0,\varepsilon)}(|p|^2)) |\hat{\phi}(p)|^2 \, d\varepsilon dp. \end{aligned}$$

Lemma 2

Lemma

For any sequence $\{\phi_j\}_j \subset L^2(\mathbb{R}^d)$

$$(\sum_j |\phi_j(x) - \phi_j^\varepsilon(x)|^2)^{1/2} \geq \left[(\sum_j |\phi_j(x)|^2)^{1/2} - (\sum_j |\phi_j^\varepsilon(x)|^2)^{1/2} \right]_+ ,$$

where $[f]_+$ *denotes the positive part of* f.

$$(\sum_j |\phi_j(x)|^2)^{1/2} \leq (\sum_j |\phi_j(x) - \phi_j^\varepsilon(x)|^2)^{1/2} + (\sum_j |\phi_j^\varepsilon(x)|^2)^{1/2}$$

Lemma 3

Lemma

Let $\{\phi_j\}_j$ *be an orthonormal system in* $L^2(\mathbb{R}^d)$*. Then*

$$\sum_j |\phi_j^\varepsilon(x)|^2 \leq (2\pi)^{-d} d^{-1} |S^{d-1}| \varepsilon^{d/2},$$

where $|S^{d-1}|$ *denotes the surface area of the unit ball in* $\mathbb{R}^d$*.*

For any sequence $\{\phi_j\}_j \subset L^2(\mathbb{R}^d)$, we call $\{\phi_j\}_j$ is an orthonormal system in $L^2(\mathbb{R}^d)$ if $(\phi_j, \phi_k) = \delta_{jk}$ for any j and k, where

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}.$$

We begin with

$$\phi_j^\varepsilon(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot p} \hat{\phi}_j^\varepsilon(p)\ dp.$$

Since $\{\hat{\phi}_j\}_j$ is also an orthonormal system in momentum space by assumption, we obtain

$$\begin{aligned} \sum_j |\phi_j^\varepsilon(x)|^2 &= (2\pi)^{-d} \sum_j \left| \int_{\mathbb{R}^d} e^{ix\cdot p} \hat{\phi}_j^\varepsilon(p)\ dp \right|^2 \\ &= (2\pi)^{-d} \sum_j \left| \int_{\mathbb{R}^d} e^{ix\cdot p} \chi_{[0,\varepsilon)}(|p|^2) \hat{\phi}_j(p)\ dp \right|^2 \\ &\le (2\pi)^{-d} \int_{\mathbb{R}^d} \left| e^{ix\cdot p} \chi_{[0,\varepsilon)}(|p|^2) \right|^2\ dp \end{aligned}$$

by Bessel's inequality.

At the last integral, we change variables $p = r\omega$ with $r \in [0, \infty)$ and $\omega \in S^{d-1} = \left\{\omega \in \mathbb{R}^d; |\omega| = 1\right\}$, then $dp = r^{d-1} dr d\omega$ and

$$\int_{\mathbb{R}^d} \left| e^{ix\cdot p} \chi_{[0,\varepsilon)}(|p|^2) \right|^2 \, dp = \int_{S^{d-1}} d\omega \int_0^\infty \left| e^{ix\cdot p} \chi_{[0,\varepsilon)}(r^2) \right|^2 \, r^{d-1} \, dr.$$

By the definition of χ, we have

$$\int_0^\infty \left| e^{ix\cdot p} \chi_{[0,\varepsilon)}(|p|^2) \right|^2 \, r^{d-1} dr = \int_0^{\varepsilon^{1/2}} \, r^{d-1} \, dr = d^{-1} \varepsilon^{d/2}.$$

Estimate for Kinetic Energy

We need to estimate the lower bound of kinetic energies for orthonormal systems in order to estimate the Riesz mean.

Proposition (Solovej, 2011)

Let $\{\phi_j\}_j$ be an orthonormal system in $L^2(\mathbb{R}^d)$. Then

$$\sum_j \int_{\mathbb{R}^d} |\nabla \phi_j(x)|^2 \, dx \geq \frac{(2\pi)^2 d^{2+2/d} |S^{d-1}|^{-2/d}}{(d+2)(d+4)} \int_{\mathbb{R}^d} \Big(\sum_j |\phi_j(x)|^2\Big)^{1+2/d} \, dx. \tag{2}$$

Solovej's approach

$$\int_{\mathbb{R}^d} \int_0^\infty \left[(\sum_j |\phi_j(x)|^2)^{1/2} - (\sum_j |\phi_j^\varepsilon(x)|^2)^{1/2} \right]_+^2 \, d\varepsilon dx$$
$$\geq \int_{\mathbb{R}^d} \int_0^\infty \left[A_0(x) - B\varepsilon^{d/4} \right]_+^2 \, d\varepsilon dx.$$

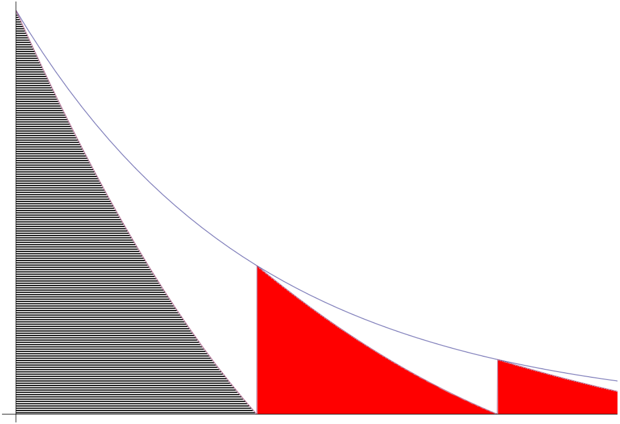
In the last inequality, we have used

$$A_0(x) = \left(\sum_j |\phi_j(x)|^2 \right)^{1/2}$$

and

$$B = (2\pi)^{-d/2} d^{-1/2} |S^{d-1}|^{1/2}.$$

Figure



Our new approach

Instead of the estimate (22), we can use

$$\int_0^\infty \left[(\sum_j |\phi_j(x)|^2)^{1/2} - (\sum_j |\phi_j^\varepsilon(x)|^2)^{1/2} \right]_+^2 \, d\varepsilon$$
$$\geq \sum_{n=0}^\infty \int_{\varepsilon_n}^{\varepsilon_{n+1}} \{A_n(x) - B(\varepsilon - \varepsilon_n)^{d/4}\}^2 \, d\varepsilon,$$

where

$$A_n(x) = A_0(x) - \left(\sum_j |\phi_j^{\varepsilon_n}(x)|^2 \right)^{1/2}$$

and $\varepsilon_{n+1} - \varepsilon_n = (A_n/B)^{4/d}$ with $\varepsilon_0 = 0$.

Applications

Lieb-Thirring inequalities can be applied

- to estimate the ground state energy of quantum systems Lieb and Thrring (1976)
- to estimate dimensions of attractors in theory of the Navier-Stokes equations Lieb (1984)
- to prove a geometrical problem for ovals in the plane Benguria and Loss (2004)

Concluding remarks

- Our new approach may improve the Lieb-Thirring coefficient with additional conditions.
- We need an example of the potential V which attains the sufficient condition.
- At the present time, Lieb-Thirring conjecture remains open.