Recent progress in Lieb-Thirring inequalities

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Abstract

Let

$$H = -\triangle + V(x) \quad \text{on } L^2(\mathbb{R}^d),$$

where $\triangle$ denotes the Laplace operator in $\mathbb{R}^d$ and V is a real-valued potential function of $x \in \mathbb{R}^d$. $E_0, E_1, E_2, \ldots$ denote all non-positive eigenvalues of H. The Riesz mean

$$\sum_j |E_j|^\gamma$$

should be estimated in the view of V.

Theorem 1 (Lieb and Thirring[9], 1976). *Let $\gamma \geq 0$. Assume that $V_-(x) = \max\{-V(x), 0\}$ satisfies the condition $V_- \in L^{\gamma+d/2}(\mathbb{R}^d)$. Then, there is $L_{\gamma,d} > 0$ which is independent of V such that*

$$\sum_j |E_j|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-(x)^{\gamma+d/2} \, dx \tag{1}$$

holds when $\gamma \geq 1/2$ for $d = 1$, $\gamma > 0$ for $d = 2$, and $\gamma \geq 0$ for $d \geq 3$. Otherwise, there is V that violates the inequalities (1) for any finite choice of $L_{\gamma,d}$.

- Lieb and Thirring [9]; almost all cases in Lieb-Thirring inequalities. (1976)
- Cwikel[1], Lieb[7] and Rozenbljum[10]; the critical case $\gamma = 0$ for $d \geq 3$. (1970s)
- Weidl[13]; the remaining case $\gamma = 1/2$ for $d = 1$. (1996)

The coefficients $L_{\gamma,d}$ should be compared to classical ones obtained by the semi-classical approximation

$$L^{\mathrm{cl}}_{\gamma,d} = (2\pi)^d \int_{|p|\leq 1} (1 - |p|^2)^\gamma \, dp = \frac{\Gamma(\gamma+1)}{(4\pi)^{d/2}\Gamma(\gamma+1+d/2)}$$

where Γ is the Gamma function. It is also known that $L_{\gamma,d}/L^{\mathrm{cl}}_{\gamma,d} \geq 1$ for all possible γ and d, and that $L_{\gamma,d}/L^{\mathrm{cl}}_{\gamma,d}$ is non-increasing on γ.

- Helffer and Robert[3]; $L_{\gamma,d}/L^{\mathrm{cl}}_{\gamma,d} > 1$ for $\gamma < 1$. (1990)
- Hundertmark, Lieb and Thomas[5]; $L_{1/2,1} = 2L^{\mathrm{cl}}_{1/2,1}$. (1998)
- Hundertmark, Laptev and Weidl[4]; $L_{\gamma,1} = L^{\mathrm{cl}}_{\gamma,1}$ for $\gamma \geq 3/2$. (2000)
- Laptev and Weidl[6]; $L_{\gamma,d} = L^{\mathrm{cl}}_{\gamma,d}$ if $\gamma \geq 3/2$ for all d. (2000)

Dolbeault, Laptev and Loss [2] have improved the coefficient.

Recent result. *Dolbeault, Laptev and Loss, 2008*

$$L_{1,d}/L^{cl}_{1,d} \leq \pi/\sqrt{3} = 1.81... \quad \text{for all } d.$$

Rumin[11] and Solovej[12] has proposed a new approach of proving that

Recent result. *Rumin and Solovej, 2011*

$$L_{1,d}/L^{cl}_{1,d} \leq \left(\frac{d+4}{d}\right)^{d/2}.$$

Conjecture (Lieb and Thirring[9], 1976).

$$L_{1,3} = L^{cl}_{1,3}$$

Collorary (Stability of Matter [8]). *Let* $Z_{max} = \max_j\{Z_j\}$. *For all normalized, antisymmetric wavefunction* ψ *with* q *spin states,*

$$(\psi, H\psi) \geq -0.747\alpha^2 N q^{2/3}\left(1 + 2.56 Z_{max}(M/N)^{1/3}\right)^2 .$$

For $\phi \in L^2(\mathbb{R}^d)$ we use

$$\phi^\varepsilon(x) = \mathcal{F}^{-1}\left[\chi_{[0,\varepsilon)}(|p|^2)\hat{\phi}(p)\right],$$

where $\mathcal{F}^{-1}$ is the Fourier inverse transform.

Condition 1 (sufficient). *Let* ϕ_j *be the eigenfunction corresponding to the eigenvalue* E_j *of* H. *Then, there is* $\kappa > 0$ *independent of* x *and* n *such that*

$$(1-\kappa^n)^2 \sum_j |\phi_j^{\varepsilon_n}(x)|^2 \leq \sum_j |\phi_j(x)|^2 \quad a.\ e.,$$

where $\varepsilon_n \geq 0$ *are some increasing sequence.*

Theorem 2. *If the condition holds, we can improve the estimate of the coefficients*

$$L_{1,d}/L_{1,d}^{cl} \leq \left(\frac{(d+4)(1-\tilde{\kappa})}{d}\right)^{d/2} \quad \text{with } 0 \leq \tilde{\kappa} \leq 4/(d+4).$$

References

[1] M. Cwikel "Weak Type Estimates for Singular Values and the Number of Bound States of Schrödinger Operators," Ann. of Math., Second Series, **106**, 93–100 (1977).

[2] J. Dolbeault, A. Laptev and M. Loss; "Lieb-Thirring inequalities with improved constants," J. Eur. Math. Soc., Vol. 10, pp. 1121-1126 (2008).

[3] B. Helffer and D. Robert, "Riesz means of bound states and semiclassical limit connected with a Lieb-Thirring's conjecture," Asymptotic Anal. **3**, no. 2, 91–103 (1990); "Riesz means of bounded states and semi-classical limit connected with a Lieb-Thirring conjecture. II," Ann. Inst. H. Poincaré Phys. Théor. **53**, no. 2, 139-147 (1990).

[4] D. Hundertmark, A. Laptev and T. Weidl, "New bounds on the Lieb-Thirring constants," Invent. Math., **140**, no. 3, 693–704 (2000).

[5] D. Hundertmark, E. H. Lieb and L. E. Thomas, "A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator," Adv. Theor. Math. Phys., **2**, no. 4, 719–731 (1998).

[6] A. Laptev and T. Weidl, "Sharp Lieb-Thirring inequalities in high dimensions," Acta Math., **184**, no. 1, 87–111 (2000).

[7] E. H. Lieb, "The number of bound states of one-body Schrödinger operators and the Weyl problem," Proc. Amer. Math. Soc. Symposia in Pure Math., **36**, 241-252 (1980).

[8] E. H. Lieb and W. E. Thirring, "Bound for the Kinetic Energy of Fermions which Proves the Stability of Matter," Phys. Rev. Lett., **35**, 687–689 (1975); Errata ibid., 1116 (1975).

[9] E. H. Lieb and W. E. Thirring, "Inequalities for the Moments of the Eigenvalues of the Schrödinger Hamiltonian and their Relation to Sobolev Inequarities," in *Studies in Mathematical Physics*, edited by E. Lieb, B. Simon and A. Wightman (Princeton University Press, Princeton, NJ, 1976), pp. 269–303.

[10] G. V. Rozenbljum, "Distribution of the discrete spectrum of singular differential operators," (Russian) Dokl. Akad. Nauk SSSR **202**, 1012–1015 (1972); Izv. VUZov, Mathematika, **1**, 75–86 (1976).

[11] M. Rumin, "Balanced distribution-energy inequalities and related entropy bounds," Duke Math. J., **160**, no. 3, 567–597 (2011).

[12] J. P. Solovej, "The Lieb-Thirring inequality," `http://www.ucl.ac.uk/~ucahipe/solovej-lt.pdf` (2011).

[13] T. Weidl, "On the Lieb-Thirring constants $L_{\gamma,1}$ for $\gamma \geq 1/2$," Comm. Math. Phys. **178**, no. 1, 135–146 (1996).